NONLINEAR DYNAMICS OF CABLE GALLOPING VIA A TWO-DEGREE-OF-FREEDOM NONLINEAR OSCILLATOR

by

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TITLE: NONLINEAR DYNAMICS OF CABLE GALLOPING VIA A TWO-DEGREE-OF-FREEDOM NONLINEAR OSCILLATOR

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The galloping vibrations of a single transmission cable that may vibrate transversely and torsionally has been investigated via a two-degree-of-freedom oscillator. The analytical solutions of periodic motions for this two-degree-of-freedom system are represented by the finite Fourier series. The analytical bifurcation trees of periodic motions to chaos of a transmission line under both steady and unsteady flows are discussed from the generalized harmonic balance method. The analytical solutions for stable and unstable periodic motions in such a two degree-of-freedom system are achieved, and the corresponding stability and bifurcation was discussed. The limit cycle for the linear cable structure are obtained by gradually decreasing the sinusoidal excitation amplitude. In addition, the numerical simulations of stable and unstable periodic motions are illustrated. The rich dynamical behavior in such a nonlinear cable structure are discovered, and this investigation may help one better understand the galloping phenomena for any elastic structures.

DEDICATION

I would like to dedicate this dissertation to my wife. Thanks to her company during the past few years. We have experienced so many things together. Without your encouragement and support, the whole process would be more difficult. I also want to express my deep gratitude to my parents who always stand behind me no matter what happens. I felt steady and strong whenever I was in a tough situation. Special dedication to my aunt who has been hospitalized for almost four months. How I wish I could be with you and take care of you. I hope someday you will come back to yourself and reunite with me.

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INTRODUCTION

1.1 Background and motivation

Flow-induced vibrations have been studied since the early 19^{th} century. In recent years, with the rapid development in engineering technology, the materials of structures (aircraft, bridge, power transmission lines) tend to become more flexible and lighter. Under the circumstances, flow-induced vibrations have been considered as one of the important factors during the design process.

When an elastic structure oscillates in a steady flow, the flow around the structure will in turn oscillate relative to the structure. The fluid forces caused by this flow tend to increase the amplitude of the oscillations. Such a phenomenon is called galloping vibrations which are one type of flow-induced vibrations. The galloping of ice-coated transmission lines in a steady wind is caused by its unsymmetrical non-circular cross section. In these situations, just like an airfoil, the vertical component of the aerodynamic force tends to increase the amplitude of the vibration in the vertical direction. The galloping vibrations of iced-cold power transmission cables are observed in Canada and North America. The vibrations not only can cause wear and damage to the transmission line hardware but also lead to severe disruptions in the electrical power supply.

To understand the mechanisms of these phenomena, different mathematical models and techniques have been applied by various researchers. However due to the complexity of fluid forces, depending on orientation and velocity of the structure, and linear modeling of most structures, many of the characters are not yet totally comprehend. In this dissertation, a linear model for the conductor is used to investigate the galloping vibrations by using the generalized harmonic balanced method. The dynamics of the linear cable under both aerodynamic and oscillating loads are studied. In addition to the linear structure, a nonlinear model is also used. Through such a nonlinear system, different dynamical response can be achieved from the linear system.

1.2 Introduction

The galloping vibrations of transmission lines have been investigated extensively since the early 1930. In the early stage, the analysis of galloping vibration was mainly based on a single-degree-of-freedom (SOF) system (Den Hartog, 1932 and Parkinson, 1989, Blevins, 1977). Later on, a two-degree-of-freedom (2DOF) model was used to study the galloping phenomena by Blevins and Nigol (Blevins, 1974, 1977; Nigol and Buchan 1981, and Richardson, 1981). Compared with the SDOF system, the 2DOF model considers the torsional effects of transmission lines, that was actually observed in field observations (Edwards and Madeyski, 1956). In addition, it has been shown that theoretically the twist motion has also played an important role for the initiation of galloping vibrations (Blevins and Iwan, 1974; Desai and Popplwell. N., 1990). In 1993, P. Yu (P. Yu, Shah and Popplewell, 1993) developed a three degree-of-freedom model. Based on his model, the galloping behavior in the plunge, twist and horizontal directions (long-wind direction) was discussed and explicit expressions for the periodic and quasi-periodic solutions of galloping were obtained. Assuming most transmission lines are deforming linear with increasing wind load, in general, the conductors are modeled as linear oscillators. However, in fact, the nonlinearities of the structures (materials and geometries) are the important factors might affect the predictions of dynamic response. So nowadays it is trendy to study the nonlinear phenomena of galloping vibration.

The modelling fluid force on the structures are very difficult and challenging. It is actually impossible to obtain a true model for the fluid forces on different bluff structures. If the oscillation of a structure is small enough, the aerodynamic force may be modeled as linear function of angle attack (Eg. airfoils). However, in most cases, the aerodynamic forces are nonlinear and coupled with the structure. Especially for the aerodynamic loads on the transmission lines, the fluid flow is separated by the structure's cross section, the fluid force is a nonlinear function of angle of attack. Generally, the fluid models depend on nonlinear curve fitting to the experimental data that are measured from the wind tunnel test. The aerodynamic force on the bluff structures can be written as a polynomial. In 1959, Slate (Slate, 1959) studied the nonlinear aerodynamic force by employing a polynomial of order as high as 25. However, such a model usually demands a heavy computation load. For all the previous models of transmission cables, dynamic responses of the transmission lines are talked about only under the aerodynamic forces. In this dissertation, the dynamics of the power transmission cables under both aerodynamic loads and external forces are investigated. The external forces are in the form of sinusoidal waves. Based on this model, different kinds of periodic vibrations can be obtained analytically for a specific set of parameters.

1.3 Mathematical techniques

For the computations of mathematical models, most research work are relying on the numerical results. However, the transmission lines are naturally slight damped, so the conventional time-marching techniques are time consuming. Even for a single degree-of-freedom model, the finding of possible steady-state galloping vibration can still be protracted. Since the models are also nonlinear, bad initial conditions can yield no galloping motions that might be actually exist. So some analytical approaches have applied to find the steady-state solutions such as Krylov-Bogoliubov time averaging method (Desai et al, 1990), the harmonic balance method (Parkinson, 1989), and multiple scale method (Nayfeh, 1981). Some of these results were generated based on the concern to estimate maximum galloping amplitude (Blevins and Iwan, 1974). Therefore, they are very crude and insufficient. Meanwhile the analytical solutions for periodic solutions are obtained only for the ratio of any two natural frequencies is close to a ratio of two positive integers (Blevins and Iwan, 1974; Desai el ta, 1990). It is known that galloping vibrations can normally have limit cycles (Blevins, 1974, 1977). Herein, one of the goals in this dissertation is to obtain the limit cycles in a 2DOF cable system.

In 2012, Luo systematically developed the generalized harmonic balanced method to obtain the analytical solutions of periodic motions and chaos in nonlinear dynamical systems. This method employs the finite Fourier series with time varying coefficients. Based on the idea of principal of virtual work, a dynamical system of coefficients is generated. Through such a dynamical system, the steady-state solution is obtained, and the corresponding stability and bifurcation are completed. Luo and Huang (2012-2014) used the generalized harmonic balanced method to study periodic motions of Duffing oscillator. The analytical bifurcation trees from period-1 motions to chaos were obtained. Then Luo and Yu (2013-2015) studied the periodic motions in a quadratic nonlinear oscillator with a periodic excitation using the generalized harmonic balanced method. The analytical routes of periodic motions to chaos are clearly presented in their work. Later on, Luo and Yu (2014-2016) also researched the dynamics of a parametric nonlinear system through the generalized harmonic balanced method. Some interesting phenomenon about the parametric systems can be discovered by applying this method. The periodic motions in a periodically forced, van del Pol oscillator was studied by Luo and Lakeh (2013). And Luo and Jin (2014) investigated the analytical solutions of periodic motions in a time-delay system. So far that's all the work that have been done for the single degree of freedom system. In 2014, Huang and Luo (2015, 2016) studied the analytical solutions of periodic motions in a Jeffcott rotor system via a two-degree-of-freedom nonlinear system. For a better understanding of periodic motions in two-degree-of-freedom nonlinear systems, Yu and Luo (2015, 2016) investigated the periodic motions in two-degree-of freedom spring mass damper systems with a nonlinear spring. In his work, the approximate analytical solutions of periodic motions for a two-degree-of-freedom nonlinear oscillators are presented, which is much more accurate than existing solutions based on perturbation and modal analysis. The accuracies of the solutions can be controlled for different parameters by choosing the number of harmonic terms. It is very helpful to use this method for some practical applications. For example, one can use the results for this two-degree-of-freedom system to understand the dynamical behavior of a tuned-mass-damper system (R. Viguie, 2009), which are beneficial for the design to avoid any unwanted interactions.

1.4 Organization of dissertation

In this dissertation, the generalized harmonic balance method is introduced in Chapter2 for nonlinear dynamical systems. The mathematical model for a single power transmission line under aerodynamics loads are derived in Chapter 3. In Chapter 4, the periodic motions in a linear cable structure under steady and unsteady flow are studied by the generalized harmonic balanced method. The bifurcation scenarios are obtained by for a specific different sets of parameters, and the galloping vibrations for this line cable structure is completed. The analytical solutions of periodic motions in a nonlinear cable structure is investigated in the Chapter 5. The bifurcation trees of period-1 motions to chaos are presented. Finally, Chapter 5 concludes this thesis and discusses the future work.

CHAPTER 2

METHODOLODY

Consider a nonlinear dynamical system as

$$\ddot{\mathbf{x}} + \mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, \mathrm{t}) = \mathbf{0} \tag{2.1}$$

where $\mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, \mathrm{t})$ is a nonlinear function vector and is periodic for time with $T = 2\pi / \Omega$.

Assume an approximate generalized periodic solution for the steady-state motion of Equation (2.1) is the form of

$$\mathbf{x}^*(t) = \mathbf{a}_0^{(m)}(t) + \sum_{k=1}^{N} \mathbf{b}_{k/m}(t)\cos(\frac{k\Omega}{m}t) + \mathbf{c}_{k/m}(t)\sin(\frac{k\Omega}{m}t) \tag{2.2}$$

Then the first and second order derivatives of $\mathbf{x}^*(t)$ are

$$\dot{\mathbf{x}}^*(t) = \dot{\mathbf{a}}_0^{(m)}(t) + \sum_{k=1}^{N} (\dot{\mathbf{b}}_{k/m} + \frac{k\Omega}{m}\mathbf{c}_{k/m})\cos(\frac{k\Omega}{m}t) + (\dot{\mathbf{c}}_{k/m} - \frac{k\Omega}{m}\mathbf{b}_{k/m})\sin(\frac{k\Omega}{m}t) \tag{2.3}$$

$$\begin{aligned}\ddot{\mathbf{x}}^*(t) = \ddot{\mathbf{a}}_0^{(m)}(t) + \sum_{k=1}^{N} (\ddot{\mathbf{b}}_{k/m} + 2\frac{k\Omega}{m}\dot{\mathbf{c}}_{k/m} - (\frac{k\Omega}{m})^2\mathbf{b}_{k/m})\cos(\frac{k\Omega}{m}t) \\ + (\ddot{\mathbf{c}}_{k/m} - 2\frac{k\Omega}{m}\dot{\mathbf{b}}_{k/m} - (\frac{k\Omega}{m})^2\mathbf{c}_{k/m})\sin(\frac{k\Omega}{m}t)\end{aligned} \tag{2.4}$$

Suppose that $\mathbf{a}_0^{(m)}(t), \mathbf{b}_{k/m}(t), \mathbf{c}_{k/m}(t)$ vary slowly with time. Substitution of Equation (2.2)-(2.4) into Equation (1) and averaging for each harmonic terms of $\cos(\frac{k\Omega}{m}t)$ and $\sin(\frac{k\Omega}{m}t)$ ($k = 1, 2, \cdots$) gives

$$\begin{aligned}&\ddot{\mathbf{a}}_0^{(m)} + \mathbf{F}_0^{(m)}(a_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{a}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) = 0, \\ &\ddot{\mathbf{b}}_{k/m} + 2\frac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{c}}_{k/m} - (\frac{\mathbf{k}_2\Omega}{m})^2\mathbf{b}_{k/m} + \mathbf{F}_{1k}^{(m)}(a_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{a}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) = 0, \\ &\ddot{\mathbf{c}}_{k/m} - 2\frac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{b}}_{k/m} - (\frac{\mathbf{k}_2\Omega}{m})^2\mathbf{c}_{k/m} + \mathbf{F}_{2k}^{(m)}(a_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{a}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) = 0 \\ &\text{for } k = 1, 2, \cdots, N\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&\mathbf{k}_1 = diag(\mathbf{I}_{n\times n}, 2\mathbf{I}_{n\times n}, \cdots, N\mathbf{I}_{n\times n}) \\
&\mathbf{k}_2 = diag(\mathbf{I}_{n\times n}, 2^2\mathbf{I}_{n\times n}, \cdots, N^2\mathbf{I}_{n\times n}) \\
&\mathbf{b}^{(m)} = (\mathbf{b}_1^{(m)}, \mathbf{b}_2^{(m)}, ..., \mathbf{b}_N^{(m)})^{\mathbf{T}}\ \mathbf{c}^{(m)} = (\mathbf{c}_1^{(m)}, \mathbf{c}_2^{(m)}, ..., \mathbf{c}_N^{(m)})^{\mathbf{T}} \\
&\mathbf{F}_1^{(m)} = (\mathbf{F}_{11}^{(m)}, \mathbf{F}_{12}^{(m)}, \cdots, \mathbf{F}_{1N}^{(m)})^{\mathrm{T}}, \\
&\mathbf{F}_2^{(m)} = (\mathbf{F}_{21}^{(m)}, \mathbf{F}_{22}^{(m)}, \cdots, \mathbf{F}_{2N}^{(m)})^{\mathrm{T}} \\
&\text{for } N = 1, 2, \cdots, \infty.
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\mathbf{F}_0^{(m)}(a_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{a}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) &= \frac{1}{mT}\int_0^{mT} \mathbf{f}(x, \dot{x}, t)dt \\
\mathbf{F}_{1k}^{(m)}(a_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{a}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) &= \frac{2}{mT}\int_0^{mT} \mathbf{f}(x, \dot{x}, t)\cos(\frac{k}{m}\Omega t)dt \\
\mathbf{F}_{2k}^{(m)}(a_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}, \dot{a}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}) &= \frac{2}{mT}\int_0^{mT} \mathbf{f}(x, \dot{x}, t)\sin(\frac{k}{m}\Omega t)dt
\end{aligned} \tag{2.7}$$

Without the assumption of slow varying with time, the averaging cannot be done for the dynamical system in Eq. (2.1) with the approximate solutions. The approximate solutions in Eq. (2.2) is treated as a transformation, form in Eq. (2.2) in an approximate solution for ready-state motion in Eq. (2.1)

Setting

$\mathbf{x} = (\mathbf{a}_0, \mathbf{b}, \mathbf{c})^T$ and $\dot{\mathbf{x}} = \mathbf{x}_1$, one obtains

$$\mathbf{g} = (-\mathbf{F}_0^{(m)}, -\mathbf{F}_{1k}^{(m)} - 2\frac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{c}}_{k/m} + (\frac{\mathbf{k}_2\Omega}{m})^2\mathbf{b}_{k/m}, -\mathbf{F}_{2k}^{(m)} + 2\frac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{b}}_{k/m} + (\frac{\mathbf{k}_2\Omega}{m})^2\mathbf{c}_{k/m})^T \tag{2.8}$$

Equation (2.5) becomes

$$\dot{\mathbf{x}} = \mathbf{x}_1 \quad \text{and} \quad \dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}, \mathbf{x}_1) \tag{2.9}$$

If $\mathbf{x}_1 = \mathbf{0}$, the equilibrium points is given by $\mathbf{g}(\mathbf{z}^*, \mathbf{0}) = \mathbf{0}_{1\times n(2N+1)}$, i.e.

$$
\begin{aligned}
&\mathbf{F}_0^{(m)}(a_0^*, \mathbf{b}^*, \mathbf{c}^*, \mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0}, \\
&\mathbf{F}_{1k}^{(m)}(a_0^*, \mathbf{b}^*, \mathbf{c}^*, \mathbf{0}, \mathbf{0}, \mathbf{0}) - (\frac{\mathbf{k}_2\Omega}{m})^2 \mathbf{b}_{k/m} = \mathbf{0}, \\
&\mathbf{F}_{2k}^{(m)}(a_0^*, \mathbf{b}^*, \mathbf{c}^*, \mathbf{0}, \mathbf{0}, \mathbf{0}) - (\frac{\mathbf{k}_2\Omega}{m})^2 \mathbf{c}_{k/m} = \mathbf{0}. \\
&\text{for } k = 1, 2, \cdots, N
\end{aligned}
\tag{2.10}
$$

The foregoing equation is given by the traditional harmonic balance method. Once the equilibrium point of $\mathbf{z}^* = (\mathbf{a}_0^*, \mathbf{b}^*, \mathbf{c}^*)^T$ is obtained, the approximate solutions in Eq. (2.2) is obtained, which gives the steady-state solution of dynamical systems in Eq. (2.1). The stability of approximate solution can be determined from Eq. (2.9). Let $\mathbf{y} = (\mathbf{z}, \mathbf{z}_1)^T$ and $\mathbf{f} = (\mathbf{z}_1, \mathbf{g})^T$. Equation (2.9) becomes $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$. The linearized equation at the equilibrium point $\mathbf{y}^* = (\mathbf{z}^*, \mathbf{0})^T$ is given by

$$\Delta\dot{\mathbf{y}}^{(m)} = D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)})\Delta\mathbf{y}^{(m)} \tag{2.11}$$

where,

$$D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)}) = \partial\mathbf{f}^{(m)}(\mathbf{y}^{(m)}) / \partial\mathbf{y}^{(m)}\Big|_{\mathbf{y}^{(m)*}} \tag{2.11}$$

The corresponding eigenvalues are determined by

$$\left| D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)}) - \lambda\mathbf{I}_{2(4N+2)\times 2(4N+2)} \right| = 0. \tag{2.12}$$

From Luo (2012), the eigenvalues of $D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)})$ are classified as

$$(n_1, n_2, n_3 \mid n_4, n_5, n_6)$$

The corresponding boundary between the stable and unstable solutions is given by the saddle-node bifurcation and Hopf bifurcation

CHAPTER 3

MECHANICAL MODEL

3.1 Mathematical model for a single transmission cable

Consider a tightly stretched cable of length l subjected to a transverse force $f(x,t)$ per unit length and an external torque $f_{\Theta}(x,t)$ per unit length, as shown in Figure 3.1 and Figure 3.2. $\theta(x,t)$ denotes the angle between the tension $N(x,t)$ and horizontal axis. c and c_{Θ} are the damping coefficient in the transverse and torsional direction respectively. $T(x,t)$ is the twisting moment. The transverse and torsional displacement are $w(x,t)$ and $\Theta(x,t)$ correspondingly. If the displacement $w(x,t)$, is assumed to be small. The application of Newton's second law yields the equations of motion:

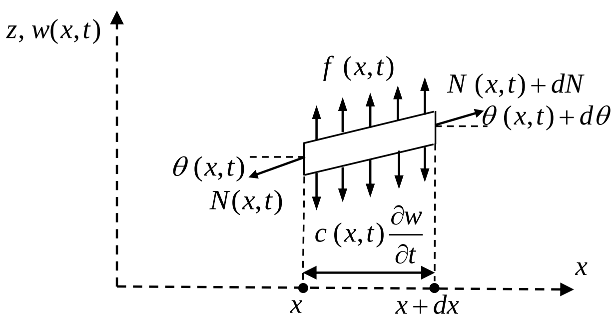


Figure 3.1: Force balance in vertical direction

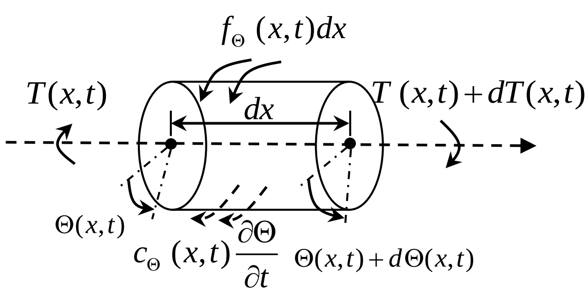


Figure 3.2: Moment balance in torsional direction

$$(N+dN)\sin(\theta+d\theta)+fdx-N\sin\theta-c(x,t)\frac{\partial w}{\partial t}dx=\rho(x)dx\frac{\partial^2 w}{\partial t^2}$$
$$(T+dT)+f_\Theta dx-T-c_\Theta(x,t)\frac{\partial\Theta}{\partial t}dx=I_0(x)dx\frac{\partial^2\Theta}{\partial t^2} \quad (3.1)$$

Where ρ is the mass per unit length and I_0 is the mass polar moment of inertia of the cable per unit length.

Let

$$\sin\theta=\frac{\partial w}{\partial x},\ \sin(\theta+d\theta)=\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial x^2}dx,\ T(x,t)=GJ(x)\frac{\partial\Theta}{\partial x}(x,t) \quad (3.2)$$

Hence the equations of motion of a uniform cable in both transverse and torsional directions can be simplified to:

$$N\frac{\partial^2 w(x,t)}{\partial x^2}+f(x,t)-c(x,t)\frac{\partial w}{\partial t}=\rho\frac{\partial^2 w}{\partial t^2}$$
$$GJ\frac{\partial^2\Theta}{\partial x^2}(x,t)+f_\Theta(x,t)-c_\Theta(x,t)\frac{\partial\Theta}{\partial t}=I_0\frac{\partial^2\Theta}{\partial t^2} \quad (3.3)$$

Which, in the case of free vibration, reduces to

$$c_1^2\frac{\partial^2 w(x,t)}{\partial x^2}=\frac{\partial^2 w}{\partial t^2}$$
$$c_2^2\frac{\partial^2\Theta}{\partial x^2}(x,t)=\frac{\partial^2\Theta}{\partial t^2} \quad (3.4)$$

Where c_1 and c_2 are:

$$c_1=\sqrt{\frac{N}{\rho}},\ c_2=\sqrt{\frac{GJ}{I_0}} \quad (3.5)$$

Eq. (3.4) are in the similar form. Both equations can be solved by using the method of separation of variable. Let

$$w(x,t)=X_1(x)T_1(t)$$
$$\Theta(x,t)=X_2(x)T_2(t) \quad (3.6)$$

Substitute of Eq.(3.6) in to Eq.(3.4) gives

$$\begin{aligned} \frac{c_1^2}{X_1}\frac{d^2X_1}{x^2} &= \frac{1}{T_1}\frac{d^2T_1}{t^2} \\ \frac{c_2^2}{X_2}\frac{d^2X_2}{x^2} &= \frac{1}{T_2}\frac{d^2T_2}{t^2} \end{aligned} \tag{3.7}$$

Since the left-hand side of this equation depends only on x and the right-hand side depends only on t, their common value must be a constant. Let

$$\begin{aligned} \frac{c_1^2}{X_1}\frac{d^2X_1}{x^2} &= \frac{1}{T_1}\frac{d^2T_1}{t^2} = -\sigma_1^2 \\ \frac{c_2^2}{X_2}\frac{d^2X_2}{x^2} &= \frac{1}{T_2}\frac{d^2T_2}{t^2} = -\sigma_2^2 \end{aligned} \tag{3.8}$$

Then the Eq.(3.8) can be written as

$$\begin{aligned} &\frac{dX_i^2(x)}{dx^2} + \frac{\sigma_i^2}{c_i^2}X_i(x) = 0 \; (i = 1, 2) \\ &\ddot{T}_i(t) + \sigma_i^2 T_i(t) = 0 \; (i = 1, 2) \end{aligned} \tag{3.9}$$

Once the solutions are obtained,

$$\begin{aligned} &X_i(x) = A_i \cos\frac{\sigma_i x}{c_i} + B_i \sin\frac{\sigma_i x}{c_i} \; (i = 1,\, 2) \\ &T_i(\mathrm{t}) = C_i \cos\sigma_i t + D_i \sin\sigma_i t \; (i = 1,\, 2) \end{aligned} \tag{3.10}$$

After applying the boundary conditions

$$X_i(0) = 0,\; X_i(l) = 0,\; (i = 1, 2) \tag{3.11}$$

The characteristic equation can be obtained

$$\frac{(\sigma_i)_n l}{c_i} = n\pi,\; (n = 1, 2, \ldots)\; (i = 1, 2) \tag{3.12}$$

Then nth the natural frequency of the problem can be written

$$(\sigma_i)_n = \frac{n\pi c_i}{l},\; (n = 1, 2, \ldots)\; (i = 1, 2) \tag{3.13}$$

The solution $w_n(x,t), \Theta_n(x,t)$ corresponding to $(\sigma_1)_n$ and $(\sigma_2)_n$ can be expressed as

$$
\begin{aligned}
w_n(x,t) &= (X_1)_n(x)(T_1)_n(t) = a_{1n}(t)\sin\frac{n\pi x}{l} \\
\Theta_n(x,t) &= (X_2)_n(x)(T_2)_n(t) = a_{2n}(t)\sin\frac{n\pi x}{l}
\end{aligned}
\tag{3.14}
$$

Where

$$
a_{in}(t) = C_{in}\cos\frac{n\pi c_i t}{l} + D_{in}\sin\frac{n\pi c_i t}{l}\ (i = 1, 2) \tag{3.15}
$$

And the total solutions can be written as

$$
\begin{aligned}
w(x,t) &= \sum_{n=1}^{\infty} w_n(x,t) = \sum_{n=1}^{\infty} a_{1n}(t)\sin\frac{n\pi x}{l} \\
\Theta(x,t) &= \sum_{n=1}^{\infty} \Theta_n(x,t) = \sum_{n=1}^{\infty} a_{2n}(t)\sin\frac{n\pi x}{l}
\end{aligned}
\tag{3.16}
$$

$w_n(x,t)$ and $\Theta_n(x,t)$ are the nth mode of the vibration in the transverse and torsional directions.

For a lightly iced transmission line, its eccentricity can be assumed negligible. Substituting $w(x,t)$ and $\Theta(x,t)$ into Eq. (3.3) and use of orthogonality of sinusoid functions

$$
\frac{2}{l}\int_0^l \xi_n(x)\xi_m(x)dx = \begin{matrix} 1,\ m = n \\ 0,\ m \neq n \end{matrix} \tag{3.17}
$$

where

$$
\xi_n(x) = \sin\frac{n\pi x}{l},\ \xi_m(x) = \sin\frac{m\pi x}{l} \tag{3.18}
$$

Lead in the usual way to the normal coordinate equations.

$$
\begin{aligned}
&\rho \ddot{a}_{1n}(t) + \sum_{m=1,2,\ldots} \dot{a}_{1m}(t) \frac{2}{l}\int_0^l c(x,t)\xi_m(x)\xi_n(x)\,\mathrm{d}\,x + N\left(\frac{n\pi}{l}\right)^2 a_{1n}(t) \\
&= \frac{2}{l}\int_0^l f(x,t)\xi_n(x)\,\mathrm{d}\,x \\
&I_0\ddot{a}_{2n}(t) + \sum_{m=1,2,\ldots} \dot{a}_{2m}(t) \frac{2}{l}\int_0^l c_\Theta(x,t)\xi_m(x)\xi_n(x)\,\mathrm{d}\,x + GJ\left(\frac{n\pi}{l}\right)^2 a_{2n}(t) \\
&= \frac{2}{l}\int_0^l f_\Theta(x,t)\xi_n(x)\,\mathrm{d}\,x
\end{aligned}
\tag{3.19}
$$

Use the new notations for $v(t) = a_{1n}(\mathrm{t})$, $\theta = a_{2n}(t)$ and let $c(x,t), c_\Theta(x,t)$ be constant.

Then substituting,

$$
\begin{aligned}
&\mathfrak{M} = \rho,\ c(x,t) = c_y,\ k_y = N\left(\frac{n\pi}{l}\right)^2,\ F_y = \frac{2}{l}\int_0^l f(x,t)\xi_n(x)\,\mathrm{d}\,x,\ a_{1n}(t) = v(t) \\
&I = I_0,\ c_\Theta(x,t) = c_\theta,\ k_\theta = GJ\left(\frac{n\pi}{l}\right)^2,\ F_M = \frac{2}{l}\int_0^l f_\Theta(x,t)\xi_n(x)\,\mathrm{d}\,x,\ a_{2n}(t) = \theta(t)
\end{aligned}
\tag{3.20}
$$

The Eq.(3.19) can be rewritten as

$$
\begin{aligned}
&\mathfrak{M}\ddot{v}(t) + c_y\dot{v}(t) + k_y v(t) = F_y \\
&I\ddot{\theta}(t) + c_\theta\dot{\theta}(t) + k_\theta\theta(t) = F_M
\end{aligned}
\tag{3.21}
$$

3.2 Mathematical model for fluid force and moment

The corresponding generalized aerodynamic load, F_y and F_M, can be expressed conventionally, as show in Figure 3.3-3.5 .

$$
\begin{aligned}
F_y &= \frac{1}{2}\rho U^2 d C_y(\alpha) \\
F_M &= \frac{1}{2}\rho U^2 d^2 C_m(\alpha)
\end{aligned}
\tag{3.22}
$$

Where, ρ is the density of the air. U is the steady wind speed. d is the conductor's diameter. $C_y(\alpha)$ and $C_m(\alpha)$ are nonlinear functions of the angle of attack α that depends on the cross section and Reynolds number.

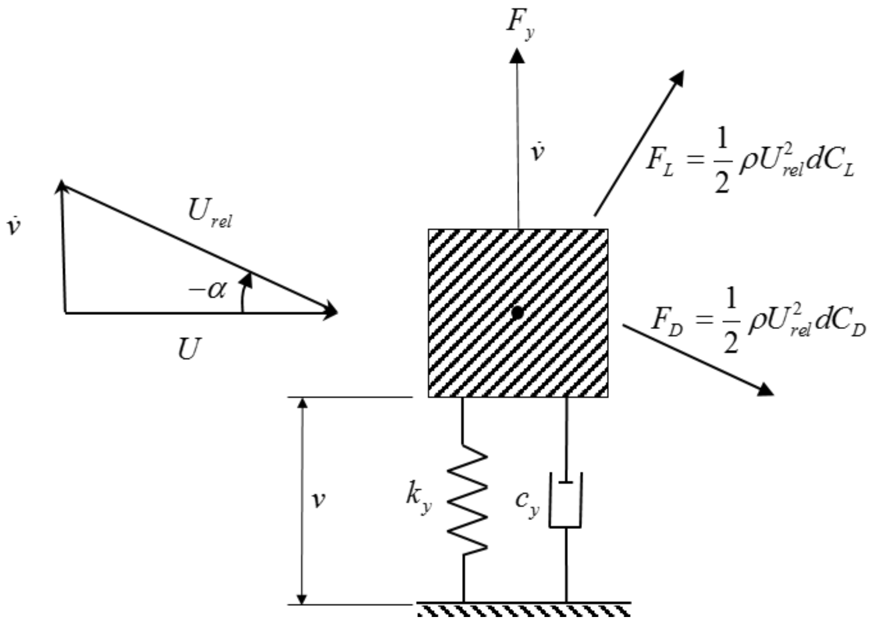


Figure 3.3: Translation

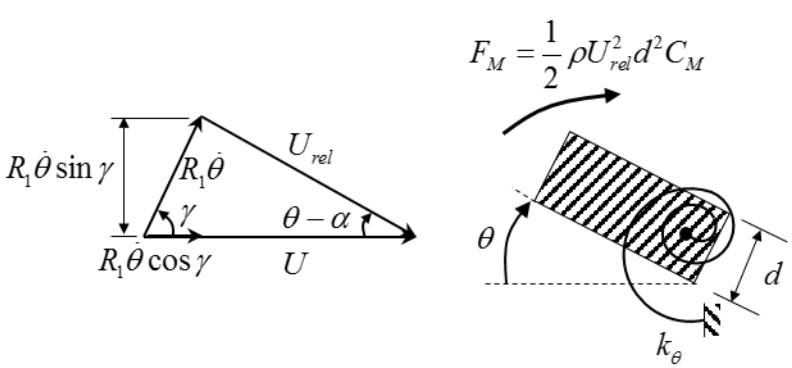


Figure 3.4: Rotation (torsional damper is not showing)

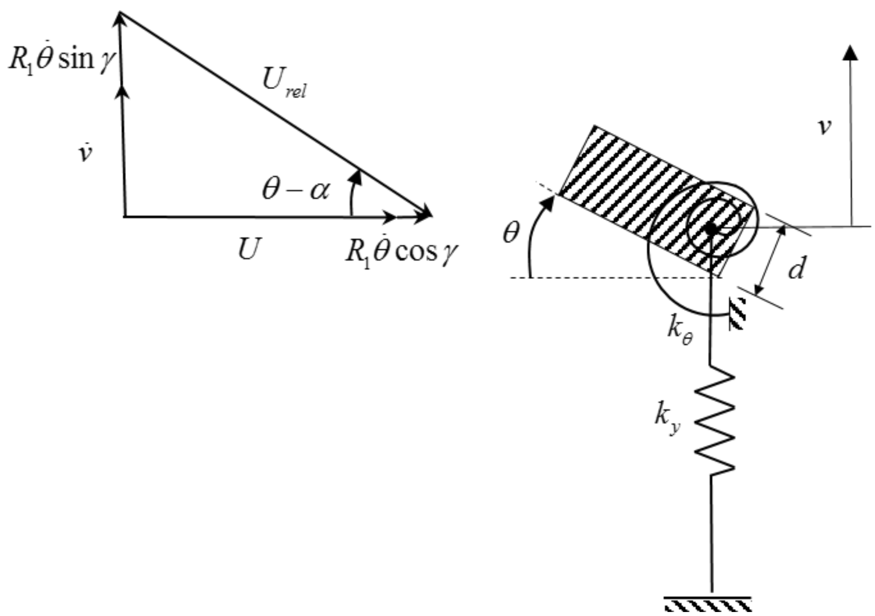


Figure 3.5: Translation and Rotation (linear and torsional damper are not showing)

The vertical force coefficient $C_y(\alpha)$ and torque coefficient are

$$C_y(\alpha) = \frac{U_{rel}^2}{U^2}(C_L\cos\alpha + C_D\sin\alpha)$$
$$C_m(\alpha) = \frac{U_{rel}^2}{U^2}C_M \tag{3.23}$$

Usually they are represented in polynomial form. Slater (1959), used 25th order polynomials to predict the solutions. However, the analytical solutions are difficult to obtain in this case. In this dissertation, the coefficients are approximated by using the cubic order polynomials. C_L, and C_D are the aerodynamic coefficient of life and drag respectively. C_M is the torque coefficient measured in the wind tunnel tests about point of rotation.

In 1974, Blevins and Iwan used a third order polynomial,

$$C_y(\alpha) = -a_1\alpha + a_3\alpha^3$$
$$C_m(\alpha) = -b_1\alpha + b_3\alpha^3 \tag{3.24}$$

a_1, a_3, b_1, b_3 are constant coefficients that can be obtained by curve-fitting experimental quasi-static wind loads.

For small angles of attack, $\alpha \ll 1$,

$$\begin{aligned} &U_{rel} \cong U \\ &\alpha \cong \theta - R\dot{\theta}/U - \dot{v}/U \end{aligned} \tag{3.25}$$

Where

$$R = R_1 \sin\gamma \tag{3.26}$$

Plug Eq.(3.22-3.26) back into Eq.(3.21). The equation of motion can be expressed as

$$\begin{aligned} &\mathfrak{M}\ddot{v}(t) + c_y\dot{v}(t) + k_y v(t) \\ &= \frac{1}{2}\rho U^2 d[-a_1(\theta - R\dot{\theta}/U - \dot{v}/U) + a_3(\theta - R\dot{\theta}/U - \dot{v}/U)^3] \\ &I\ddot{\theta}(t) + c_\theta\dot{\theta}(t) + k_\theta\theta(t) \\ &= \frac{1}{2}\rho U^2 d^2[-b_1(\theta - R\dot{\theta}/U - \dot{v}/U) + b_3(\theta - R\dot{\theta}/U - \dot{v}/U)^3] \end{aligned} \tag{3.27}$$

In this dissertation, the galloping vibration of the linear cable are first studied via such a two degree of freedom nonlinear oscillator. The analytical solutions are obtained for the limit cycle under $1:1$ case. Then periodic galloping of such a cable system with cubic nonlinear stiffness in both directions are mainly investigated. The analytical routes of periodic motion to chaos are achieved. Numerical simulations are illustrated to demonstrate the complexities of the periodic motion in both transverse and torsional directions.

The equation of motion for such a cable with nonlinear term can be written as:

$$\begin{aligned} &\mathfrak{M}\ddot{v} + c_y\dot{v} + k_y v + k_y^{'} v^3 \\ &= \frac{1}{2}\rho U^2 d[-a_1(\theta - R\dot{\theta}/U - \dot{v}/U) + a_3(\theta - R\dot{\theta}/U - \dot{v}/U)^3] \\ &I\ddot{\theta} + c_\theta\dot{\theta} + k_\theta\theta + k_\theta^{'}\theta^3 \\ &= \frac{1}{2}\rho U^2 d^2[-b_1(\theta - R\dot{\theta}/U - \dot{v}/U) + b_3(\theta - R\dot{\theta}/U - \dot{v}/U)^3] \end{aligned} \tag{3.28}$$

CHAPTER 4

ANALYTICAL SOLUTIONS FOR LINEAR CABLE STRUCTURE

In this chapter, the galloping response of a linear cable that may vibrate both transversely and torsionally are studied by using generalized harmonic balance method. Analytical solutions for periodic motions are presented in Fourier series form with finite harmonic terms, and the stability and bifurcation of the corresponding periodic motions are completed. The galloping vibrations are discussed for such a linear cable structure with a small periodic excitation. Then harmonic amplitude effect of the period-1 motion under different external periodic excitation amplitudes was investigated. The frequency of the limit cycle was also obtained. For a better understanding of the galloping vibration for such a two degree-of freedom cable system, trajectories and amplitude spectrums are illustrated.

4.1 Analytical Solutions

Consider a linear cable structure with aerodynamic load and external load.

$$\begin{aligned} &\mathfrak{M}\ddot{v}(t)+c_y\dot{v}(t)+k_yv(t)=F_y+F_e \\ &I\ddot{\theta}(t)+c_\theta\dot{\theta}(t)+k_\theta\theta(t)=F_M \end{aligned} \tag{4.1}$$

Where the aerodynamic force and moment F_y, F_M and the external load F_e can be represented as

$$\begin{aligned} &F_y=\frac{1}{2}\rho U^2 d[-a_1(\theta-R\dot{\theta}/U-\dot{v}/U)+a_3(\theta-R\dot{\theta}/U-\dot{v}/U)^3] \\ &F_M=\frac{1}{2}\rho U^2 d^2[-b_1(\theta-R\dot{\theta}/U-\dot{v}/U)+b_3(\theta-R\dot{\theta}/U-\dot{v}/U)^3] \\ &F_e=Q_0\cos\Omega t \end{aligned} \tag{4.2}$$

The standard form of Eq. (4.1) can be written as

$$\ddot{\mathbf{x}} + \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, t) = \mathbf{0} \tag{4.3}$$

Where,

$$\mathbf{x} = (v, \theta)^{\mathrm{T}}, \dot{\mathbf{x}} = (\dot{v}, \dot{\theta})^{\mathrm{T}}, \mathbf{f} = (f_1, f_2)^{\mathrm{T}} \tag{4.4}$$

$$\begin{aligned}
f_1 &= \alpha_1\dot{\theta}^3 + \alpha_2\dot{\theta}^2\dot{v} + \alpha_3\theta\dot{\theta}^2 + \alpha_4\dot{\theta}\dot{v}^2 + \alpha_5\theta\dot{\theta}\dot{v} + \alpha_6\theta^2\dot{\theta} + \alpha_7\dot{\theta} + \alpha_8\dot{v}^3 \\
&\quad + \alpha_9\theta\dot{v}^2 + \alpha_{10}\theta^2\dot{v} + \alpha_{11}\dot{v} + \alpha_{12}\theta^3 + \alpha_{13}\theta + \alpha_{14}\dot{v} + \alpha_{15}v + Q\cos\Omega t \\
f_2 &= \beta_1\dot{\theta}^3 + \beta_2\dot{\theta}^2\dot{v} + \beta_3\theta\dot{\theta}^2 + \beta_4\dot{\theta}\dot{v}^2 + \beta_5\theta\dot{\theta}\dot{v} + \beta_6\theta^2\dot{\theta} + \beta_7\dot{\theta} + \beta_8\dot{v}^3 \\
&\quad + \beta_9\theta\dot{v}^2 + \beta_{10}\theta^2\dot{v} + \beta_{11}\dot{v} + \beta_{12}\theta^3 + \beta_{13}\theta + \beta_{14}\dot{\theta} + \beta_{15}\theta
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&\alpha_1 = \frac{\rho dR^3a_3}{2U\mathfrak{M}}\ \alpha_2 = \frac{3\rho dR^2a_3}{2U\mathfrak{M}}, \alpha_3 = -\frac{3\rho dR^2a_3}{2\mathfrak{M}}, \alpha_4 = \frac{3\rho dRa_3}{2U\mathfrak{M}}, \alpha_5 = -\frac{6\rho dRa_3}{2\mathfrak{M}}, \\
&\alpha_6 = \frac{3\rho dRUa_3}{2\mathfrak{M}}, \alpha_7 = -\frac{\rho dRUa_1}{2\mathfrak{M}}, \alpha_8 = \frac{\rho da_3}{2U\mathfrak{M}}, \alpha_9 = -\frac{3\rho da_3}{2\mathfrak{M}}, \alpha_{10} = \frac{3\rho dUa_3}{2\mathfrak{M}}, \\
&\alpha_{11} = -\frac{\rho dUa_1}{2\mathfrak{M}}, \alpha_{12} = -\frac{\rho dU^2a_3}{2\mathfrak{M}}, \alpha_{13} = \frac{\rho dU^2a_1}{2\mathfrak{M}}, \alpha_{14} = \frac{c_y}{\mathfrak{M}}, \alpha_{15} = \frac{k_y}{\mathfrak{M}}, Q = -\frac{Q_0}{\mathfrak{M}} \\
&\beta_1 = \frac{\rho d^2R^3b_3}{2UI}\ \beta_2 = \frac{3\rho d^2R^2b_3}{2UI}, \beta_3 = -\frac{3\rho d^2R^2b_3}{2I}, \beta_4 = \frac{3\rho d^2Rb_3}{2UI}, \beta_5 = -\frac{6\rho d^2Rb_3}{2I}, \\
&\beta_6 = \frac{3\rho d^2RUb_3}{2I}, \beta_7 = -\frac{\rho d^2RUb_1}{2I}, \beta_8 = \frac{\rho d^2b_3}{2UI}, \beta_9 = -\frac{3\rho d^2b_3}{2I}, \beta_{10} = \frac{3\rho d^2Ub_3}{2I}, \\
&\beta_{11} = -\frac{\rho d^2Ub_1}{2I}, \beta_{12} = -\frac{\rho d^2U^2b_3}{2I}, \beta_{13} = \frac{\rho d^2U^2b_1}{2I}, \beta_{14} = \frac{c_\theta}{I}, \beta_{15} = \frac{k_\theta}{I}
\end{aligned} \tag{4.6}$$

The analytical solution of period-m motion for the above equations are

$$\begin{aligned}
v^*(t) &= a_{10}^{(m)}(t) + \sum_{k=1}^{N} b_{1k/m}(t)\cos(\frac{k\Omega t}{m}) + c_{1k/m}(t)\sin(\frac{k\Omega t}{m}), \\
\theta^*(t) &= a_{20}^{(m)}(t) + \sum_{k=1}^{N} b_{2k/m}(t)\cos(\frac{k\Omega t}{m}) + c_{2k/m}(t)\sin(\frac{k\Omega t}{m});
\end{aligned} \tag{4.7}$$

Where m is the ratio of the period of periodic motions to the excitation period.

Then the first and second order derivatives of $v^*(t)$ and $\theta^*(t)$ are

$$
\begin{aligned}
\dot{v}^*(t) &= \dot{a}_{10}^{(m)} + \sum_{k=1}^{N} (\dot{b}_{1k/m} + \frac{k\Omega}{m} c_{1k/m}) \cos(\frac{k\Omega}{m} t) + (\dot{c}_{1k/m} - \frac{k\Omega}{m} b_{1k/m}) \sin(\frac{k\Omega}{m} t), \\
&= \dot{a}_{10}^{(m)} + \sum_{k=1}^{N} P_{k/m} \cos(\frac{k\Omega}{m} t) + Q_{k/m} \sin(\frac{k\Omega}{m} t), \\
\dot{\theta}^*(t) &= \dot{a}_{20}^{(m)} + \sum_{k=1}^{N} (\dot{b}_{2k/m} + \frac{k\Omega}{m} c_{2k/m}) \cos(\frac{k\Omega}{m} t) + (\dot{c}_{2k/m} - \frac{k\Omega}{m} b_{2k/m}) \sin(\frac{k\Omega}{m} t); \\
&= \dot{a}_{20}^{(m)} + \sum_{k=1}^{N} B_{k/m} \cos(\frac{k\Omega}{m} t) + C_{k/m} \sin(\frac{k\Omega}{m} t); \\
\ddot{v}^*(t) &= \ddot{a}_{10}^{(m)} + \sum_{k=1}^{N} [\ddot{b}_{1k/m} + 2\frac{k\Omega}{m} \dot{c}_{1k/m} - (\frac{k\Omega}{m})^2 b_{1k/m}] \cos(\frac{k\Omega}{m} t) \\
&\quad + [\ddot{c}_{1k/m} - 2\frac{k\Omega}{m} \dot{b}_{1k/m} - (\frac{k\Omega}{m})^2 c_{1k/m}] \sin(\frac{k\Omega}{m} t), \\
\ddot{\theta}^*(t) &= \ddot{a}_{20}^{(m)} + \sum_{k=1}^{N} [\ddot{b}_{2k/m} + 2\frac{k\Omega}{m} \dot{c}_{2k/m} - (\frac{k\Omega}{m})^2 b_{2k/m}] \cos(\frac{k\Omega}{m} t) \\
&\quad + [\ddot{c}_{2k/m} - 2\frac{k\Omega}{m} \dot{b}_{2k/m} - (\frac{k\Omega}{m})^2 c_{2k/m}] \sin(\frac{k\Omega}{m} t).
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
B_{i/m} &= \dot{b}_{2i/m} + i\Omega c_{2i/m} / m,\ C_{i/m} = \dot{c}_{2i/m} - i\Omega b_{2i/m} / m \\
P_{i/m} &= \dot{b}_{1i/m} + i\Omega c_{1i/m} / m,\ Q_{i/m} = \dot{c}_{1i/m} - i\Omega b_{1i/m} / m
\end{aligned}
\tag{4.9}
$$

Define

$$
\begin{aligned}
\mathbf{a}_0^{(m)} &= (a_{10}^{(m)}, a_{20}^{(m)})^{\mathrm{T}}, \\
\mathbf{b}^{(m)} &= (b_{11/m}, b_{12/m}, \cdots, b_{1N/m}, b_{21/m}, b_{22/m}, \cdots, b_{2N/m})^{\mathrm{T}} \\
&= (\mathbf{b}_1^{(m)}; \mathbf{b}_2^{(m)}). \\
\mathbf{c}^{(m)} &= (c_{11/m}, c_{12/m}, \cdots, c_{1N/m}, c_{21/m}, c_{22/m}, \cdots, c_{2N/m})^{\mathrm{T}} \\
&= (\mathbf{c}_1^{(m)}; \mathbf{c}_2^{(m)}).
\end{aligned}
\tag{4.10}
$$

Substitution of Eqs.(4.8), (4.9) into Eq.(4.3) and averaging for the harmonic terms of $\cos(k\Omega t / m)$ and $\sin(k\Omega t / m)$ $(k = 0,1,2,\ldots)$ gives

$$
\begin{aligned}
&\ddot{a}_{10}^{(m)} + F_{10}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0\\
&\ddot{b}_{1k/m} + 2\frac{k\Omega}{m}\dot{c}_{1k/m} - (\frac{k\Omega}{m})^2 b_{1k/m} + F_{1k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0,\\
&\ddot{c}_{1k/m} - 2\frac{k\Omega}{m}\dot{b}_{1k/m} - (\frac{k\Omega}{m})^2 c_{1k/m} + F_{1k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0;\\
&\ddot{a}_{20}^{(m)} + F_{20}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0,\\
&\ddot{b}_{2k/m} + 2k\Omega\dot{c}_{2k/m} - (\frac{k\Omega}{m})^2 b_{2k/m} + F_{2k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0,\\
&\ddot{c}_{2k/m} - 2k\Omega\dot{b}_{2k/m} - (\frac{k\Omega}{m})^2 c_{2k/m} + F_{2k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0.
\end{aligned}
\tag{4.11}
$$

Where

$$
\begin{aligned}
F_{10}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{1}{mT}\int_0^{mT} f_1(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)dt\\
F_{1k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{2}{mT}\int_0^{mT} f_1(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)\cos(\frac{k\Omega}{m}t)dt\\
F_{1k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{2}{mT}\int_0^{mT} f_1(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)\sin(\frac{k\Omega}{m}t)dt\\
F_{20}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{1}{mT}\int_0^{mT} f_2(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)dt\\
F_{2k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{2}{mT}\int_0^{mT} f_2(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)\cos(\frac{k\Omega}{m}t)dt\\
F_{2k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{2}{mT}\int_0^{mT} f_2(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)\sin(\frac{k\Omega}{m}t)dt
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
F_{10}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\alpha_i f_1^i + \alpha_{14} f_{11}^{14} + \alpha_{15} f_{11}^{15}\\
F_{1k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\alpha_i f_2^i + \alpha_{14} f_{21}^{14} + \alpha_{15} f_{21}^{15} + Qf_1^{16}\\
F_{1k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\alpha_i f_3^i + \alpha_{14} f_{31}^{14} + \alpha_{15} f_{31}^{15}\\
F_{20}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\beta_i f_1^i + \beta_{14} f_{12}^{14} + \beta_{15} f_{12}^{15}\\
F_{2k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\beta_i f_2^i + \beta_{14} f_{22}^{14} + \beta_{15} f_{22}^{15}\\
F_{2k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\beta_i f_3^i + \beta_{14} f_{32}^{14} + \beta_{15} f_{32}^{15}
\end{aligned}
\tag{4.13}
$$

Define

$$
\begin{aligned}
\mathbf{z}^{(m)} &= (a_{10}^{(m)}, b_{11/m}, \cdots, b_{1N/m}, c_{11/m}, \cdots, c_{1N/m}; a_{20}^{(m)}, b_{21/m}, \cdots, b_{2N/m}, c_{21/m}, \cdots, c_{2N/m})^{\mathrm{T}} \\
&\equiv (z_1^{(m)}, z_2^{(m)}, \cdots, z_{2N+1}^{(m)}; z_{2N+2}^{(m)}, z_{2N+3}^{(m)}, \cdots, z_{4N+2}^{(m)})^{\mathrm{T}} \\
\mathbf{z}_1^{(m)} \triangleq \dot{\mathbf{z}}^{(m)} &= (\dot{a}_{10}^{(m)}, \dot{b}_{11/m}, \cdots, \dot{b}_{1N/m}, \dot{c}_{11/m}, \cdots, \dot{c}_{1N/m}; \dot{a}_{20}^{(m)}, \dot{b}_{21/m}, \cdots, \dot{b}_{2N/m}, \dot{c}_{21/m}, \cdots, \dot{c}_{2N/m})^{\mathrm{T}} \\
&\equiv (\dot{z}_1^{(m)}, \dot{z}_2^{(m)}, \cdots, \dot{z}_{2N+1}^{(m)}; \dot{z}_{2N+2}^{(m)}, \dot{z}_{2N+3}^{(m)}, \cdots, \dot{z}_{4N+2}^{(m)})^{\mathrm{T}}
\end{aligned}
\tag{4.14}
$$

Equations (4.11) can be rewritten as

$$
\dot{\mathbf{z}}^{(m)} = \mathbf{z}_1^{(m)} \text{ and } \dot{\mathbf{z}}_1^{(m)} = \mathbf{g}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \tag{4.15}
$$

where

$$
\mathbf{g}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) = \begin{pmatrix}
-F_{10}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \\
-\mathbf{F}_{1/m}^{(c)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) - 2\dfrac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{c}}_1^{(m)} + \mathbf{k}_2(\dfrac{\Omega}{m})^2\mathbf{b}_1^{(m)} \\
-\mathbf{F}_{1/m}^{(s)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) + 2\dfrac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{b}}_1^{(m)} + \mathbf{k}_2(\dfrac{\Omega}{m})^2\mathbf{c}_1^{(m)} \\
-F_{20}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \\
-\mathbf{F}_{2/m}^{(c)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) - 2\dfrac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{c}}_2^{(m)} + \mathbf{k}_2(\dfrac{\Omega}{m})^2\mathbf{b}_2^{(m)} \\
-\mathbf{F}_{2/m}^{(s)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) + 2\dfrac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{b}}_2^{(m)} + \mathbf{k}_2(\dfrac{\Omega}{m})^2\mathbf{c}_2^{(m)}
\end{pmatrix} \tag{4.16}
$$

where

$$
\begin{aligned}
&\mathbf{k}_1 = diag(1, 2, \cdots, N), \\
&\mathbf{k}_2 = diag(1, 2^2, \cdots, N^2), \\
&\mathbf{F}_{1/m}^{(c)} = (F_{11/m}^{(c)}, F_{12/m}^{(c)}, \cdots, F_{1N/m}^{(s)})^{\mathrm{T}}, \\
&\mathbf{F}_{1/m}^{(s)} = (F_{11/m}^{(s)}, F_{12/m}^{(s)}, \cdots, F_{1N/m}^{(s)})^{\mathrm{T}}, \\
&\mathbf{F}_{2/m}^{(c)} = (F_{21/m}^{(c)}, F_{22/m}^{(c)}, \cdots, F_{2N/m}^{(c)})^{\mathrm{T}}, \\
&\mathbf{F}_{2/m}^{(s)} = (F_{21/m}^{(s)}, F_{22/m}^{(s)}, \cdots, F_{2N/m}^{(s)})^{\mathrm{T}} \\
&\text{for } N = 1, 2, \cdots, \infty.
\end{aligned}
\tag{4.17}
$$

Setting

$$
\mathbf{y}^{(m)} \equiv (\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \text{ and } \mathbf{f}^{(m)} = (\mathbf{z}_1^{(m)}, \mathbf{g}^{(m)})^{\mathrm{T}}, \tag{4.18}
$$

Thus, equation (4.12) becomes

$$\dot{\mathbf{y}}^{(m)} = \mathbf{f}^{(m)}(\mathbf{y}^{(m)}). \tag{4.19}$$

The steady-state solutions for periodic motion can be obtained by setting $\dot{\mathbf{y}}^{(m)} = \mathbf{0}$, i.e.,

$$\begin{aligned}
&F_{10}^{(m)}(\mathbf{z}^{(m)},\mathbf{0}) = 0\\
&-\mathbf{F}_{1/m}^{(c)}(\mathbf{z}^{(m)},\mathbf{0}) + \mathbf{k}_2(\frac{\Omega}{m})^2\mathbf{b}_1^{(m)} = \mathbf{0}\\
&-\mathbf{F}_{1/m}^{(s)}(\mathbf{z}^{(m)},\mathbf{0}) + \mathbf{k}_2(\frac{\Omega}{m})^2\mathbf{c}_1^{(m)} = \mathbf{0}\\
&F_{20}^{(m)}(\mathbf{z}^{(m)},\mathbf{0}) = 0\\
&-\mathbf{F}_{2/m}^{(c)}(\mathbf{z}^{(m)},\mathbf{0}) + \mathbf{k}_2(\frac{\Omega}{m})^2\mathbf{b}_2^{(m)} = \mathbf{0}\\
&-\mathbf{F}_{2/m}^{(s)}(\mathbf{z}^{(m)},\mathbf{0}) + \mathbf{k}_2(\frac{\Omega}{m})^2\mathbf{c}_2^{(m)} = \mathbf{0}
\end{aligned} \tag{4.20}$$

The $(4N+2)$ nonlinear equations in Eq.(4.20) are solved by the Newton-Raphson method. In Luo [2012], the linearized equation at $\mathbf{y}^{(m)*} = (\mathbf{z}^{(m)*},\mathbf{0})^{\mathrm{T}}$ is

$$\Delta\dot{\mathbf{y}}^{(m)} = D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)})\Delta\mathbf{y}^{(m)} \tag{4.21}$$

where

$$D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)}) = \partial\mathbf{f}^{(m)}(\mathbf{y}^{(m)})/\partial\mathbf{y}^{(m)}\Big|_{\mathbf{y}^{(m)*}} \tag{4.22}$$

The corresponding eigenvalues are determined by

$$\left|D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)}) - \lambda\mathbf{I}_{4(2N+1)\times 4(2N+1)}\right| = 0. \tag{4.23}$$

where

$$D\mathbf{f}(\mathbf{y}^{(m)*}) = \begin{bmatrix} \mathbf{0}_{2(2N+1)\times 2(2N+1)} & \mathbf{I}_{2(2N+1)\times 2(2N+1)} \\ \mathbf{G}_{2(2N+1)\times 2(2N+1)} & \mathbf{H}_{2(2N+1)\times 2(2N+1)} \end{bmatrix} \tag{4.24}$$

and

$$\mathbf{G}=\frac{\partial \mathbf{g}^{(m)}}{\partial \mathbf{z}^{(m)}}=(\mathbf{G}^{(10)},\mathbf{G}^{(1c)},\mathbf{G}^{(1s)},\mathbf{G}^{(20)},\mathbf{G}^{(2c)},\mathbf{G}^{(2s)})^{\mathrm{T}} \tag{4.25}$$

$$\begin{aligned}\mathbf{G}^{(i0)}&=(G_0^{(i0)},G_1^{(i0)},\cdots,G_{4N+1}^{(i0)}),\\ \mathbf{G}^{(ic)}&=(\mathbf{G}_1^{(ic)},\mathbf{G}_2^{(ic)},\cdots,\mathbf{G}_N^{(ic)})^{\mathrm{T}},\\ \mathbf{G}^{(is)}&=(\mathbf{G}_1^{(is)},\mathbf{G}_2^{(is)},\cdots,\mathbf{G}_N^{(is)})^{\mathrm{T}}\end{aligned} \tag{4.26}$$

for $i=1,2;$ and $N=1,2,\cdots\infty$ with

$$\begin{aligned}\mathbf{G}_k^{(ic)}&=(G_{k0}^{(ic)},G_{k1}^{(ic)},\cdots,G_{k(4N+1)}^{(ic)}),\\ \mathbf{G}_k^{(is)}&=(G_{k0}^{(is)},G_{k1}^{(is)},\cdots,G_{k(4N+1)}^{(is)})\end{aligned} \tag{4.27}$$

for $k=1,2,\cdots N.$ The corresponding components are

$$\begin{aligned}G_r^{(10)}&=-\alpha_{15}\delta_0^r-\delta_{2N+1}^r(\alpha_3\frac{\partial f_1^3}{\partial a_{20}}+\alpha_5\frac{\partial f_1^5}{\partial a_{20}}+\alpha_6\frac{\partial f_1^6}{\partial a_{20}}+\alpha_9\frac{\partial f_1^9}{\partial a_{20}}+\alpha_{10}\frac{\partial f_1^{10}}{\partial a_{20}}+\alpha_{12}\frac{\partial f_1^{12}}{\partial a_{20}}\\&\quad+\alpha_{13})-g_r^{(10)}\\ G_{kr}^{(1c)}&=-\delta_{2N+1}^r(\alpha_3\frac{\partial f_2^3}{\partial a_{20}}+\alpha_5\frac{\partial f_2^5}{\partial a_{20}}+\alpha_6\frac{\partial f_2^6}{\partial a_{20}}+\alpha_9\frac{\partial f_2^9}{\partial a_{20}}+\alpha_{10}\frac{\partial f_2^{10}}{\partial a_{20}}+\alpha_{12}\frac{\partial f_2^{12}}{\partial a_{20}})-g_{kr}^{(1c)}\\ G_{kr}^{(1s)}&=-\delta_{2N+1}^r(\alpha_3\frac{\partial f_3^3}{\partial a_{20}}+\alpha_5\frac{\partial f_3^5}{\partial a_{20}}+\alpha_6\frac{\partial f_3^6}{\partial a_{20}}+\alpha_9\frac{\partial f_3^9}{\partial a_{20}}+\alpha_{10}\frac{\partial f_3^{10}}{\partial a_{20}}+\alpha_{12}\frac{\partial f_3^{12}}{\partial a_{20}})-g_{kr}^{(1s)}\\ G_r^{(20)}&=-\delta_{2N+1}^r(\beta_3\frac{\partial f_1^3}{\partial a_{20}}+\beta_5\frac{\partial f_1^5}{\partial a_{20}}+\beta_6\frac{\partial f_1^6}{\partial a_{20}}+\beta_9\frac{\partial f_1^9}{\partial a_{20}}+\beta_{10}\frac{\partial f_1^{10}}{\partial a_{20}}+\beta_{12}\frac{\partial f_1^{12}}{\partial a_{20}}+\beta_{13}\\&\quad+\beta_{15})-g_r^{(20)}\\ G_{kr}^{(2c)}&=-\delta_{2N+1}^r(\beta_3\frac{\partial f_2^3}{\partial a_{20}}+\beta_5\frac{\partial f_2^5}{\partial a_{20}}+\beta_6\frac{\partial f_2^6}{\partial a_{20}}+\beta_9\frac{\partial f_2^9}{\partial a_{20}}+\beta_{10}\frac{\partial f_2^{10}}{\partial a_{20}}+\beta_{12}\frac{\partial f_2^{12}}{\partial a_{20}})-g_{kr}^{(2c)}\\ G_{kr}^{(2s)}&=-\delta_{2N+1}^r(\beta_3\frac{\partial f_3^3}{\partial a_{20}}+\beta_5\frac{\partial f_3^5}{\partial a_{20}}+\beta_6\frac{\partial f_3^6}{\partial a_{20}}+\beta_9\frac{\partial f_3^9}{\partial a_{20}}+\beta_{10}\frac{\partial f_3^{10}}{\partial a_{20}}+\beta_{12}\frac{\partial f_3^{12}}{\partial a_{20}})-g_{kr}^{(2s)}\end{aligned} \tag{4.28}$$

where for $r=0,1,\cdots,4N+1$.

The derivative of the constant term for the transverse motion is

$$g_r^{(10)} = g_{r1}^{(10)} + g_{r2}^{(10)} + g_{r3}^{(10)} + g_{r4}^{(10)}, \tag{4.29}$$

with

$$\begin{aligned}
g_{r1}^{(10)} &= \sum_{n=1}^{N} \delta_n^r \frac{\partial Q_n}{\partial b_{1n}} (\alpha_2 \frac{\partial f_1^2}{\partial Q_n} + \alpha_4 \frac{\partial f_1^4}{\partial Q_n} + \alpha_5 \frac{\partial f_1^5}{\partial Q_n} + \alpha_8 \frac{\partial f_1^8}{\partial Q_n} + \alpha_9 \frac{\partial f_1^9}{\partial Q_n} \\
&\quad + \alpha_{10} \frac{\partial f_1^{10}}{\partial Q_n}) \\
g_{r2}^{(10)} &= \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial P_n}{\partial c_{1n}} (\alpha_2 \frac{\partial f_1^2}{\partial P_n} + \alpha_4 \frac{\partial f_1^4}{\partial P_n} + \alpha_5 \frac{\partial f_1^5}{\partial P_n} + \alpha_8 \frac{\partial f_1^8}{\partial P_n} + \alpha_9 \frac{\partial f_1^9}{\partial P_n} \\
&\quad + \alpha_{10} \frac{\partial f_1^{10}}{\partial P_n}) \\
g_{r3}^{(10)} &= \sum_{n=1}^{N} \delta_{n+2N+1}^r (\frac{\partial C_n}{\partial b_{2n}} (\alpha_1 \frac{\partial f_1^1}{\partial C_n} + \alpha_2 \frac{\partial f_1^2}{\partial C_n} + \alpha_3 \frac{\partial f_1^3}{\partial C_n} + \alpha_4 \frac{\partial f_1^4}{\partial C_n} + \alpha_5 \frac{\partial f_1^5}{\partial C_n} \\
&\quad + \alpha_6 \frac{\partial f_1^6}{\partial C_n}) + \alpha_3 \frac{\partial f_1^3}{\partial b_{2n}} + \alpha_5 \frac{\partial f_1^5}{\partial b_{2n}} + \alpha_6 \frac{\partial f_1^6}{\partial b_{2n}} + \alpha_9 \frac{\partial f_1^9}{\partial b_{2n}} + \alpha_{10} \frac{\partial f_1^{10}}{\partial b_{2n}} \\
&\quad + \alpha_{12} \frac{\partial f_1^{12}}{\partial b_{2n}}) \\
g_{r4}^{(10)} &= \sum_{n=1}^{N} \delta_{n+3N+1}^r (\frac{\partial B_n}{\partial c_{2n}} (\alpha_1 \frac{\partial f_1^1}{\partial B_n} + \alpha_2 \frac{\partial f_1^2}{\partial B_n} + \alpha_3 \frac{\partial f_1^3}{\partial B_n} + \alpha_4 \frac{\partial f_1^4}{\partial B_n} + \alpha_5 \frac{\partial f_1^5}{\partial B_n} \\
&\quad + \alpha_6 \frac{\partial f_1^6}{\partial B_n}) + \alpha_3 \frac{\partial f_1^3}{\partial c_{2n}} + \alpha_5 \frac{\partial f_1^5}{\partial c_{2n}} + \alpha_6 \frac{\partial f_1^6}{\partial c_{2n}} + \alpha_9 \frac{\partial f_1^9}{\partial c_{2n}} + \alpha_{10} \frac{\partial f_1^{10}}{\partial c_{2n}} \\
&\quad + \alpha_{12} \frac{\partial f_1^{12}}{\partial c_{2n}})
\end{aligned} \tag{4.30}$$

The derivative related to the cosine term for the transverse motion is

$$g_{kr}^{(1c)} = g_{kr1}^{(1c)} + g_{kr2}^{(1c)} + g_{kr3}^{(1c)} + g_{kr4}^{(1c)} \tag{4.31}$$

with

$$
\begin{aligned}
g_{kr1}^{(1c)} &= \sum_{n=1}^{N} \delta_n^r \Big(\frac{\partial Q_n}{\partial b_{1n}}\big(\alpha_2 \frac{\partial f_2^2}{\partial Q_n} + \alpha_4 \frac{\partial f_2^4}{\partial Q_n} + \alpha_5 \frac{\partial f_2^5}{\partial Q_n} + \alpha_8 \frac{\partial f_2^8}{\partial Q_n} + \alpha_9 \frac{\partial f_2^9}{\partial Q_n} + \\
&\quad \alpha_{10} \frac{\partial f_2^{10}}{\partial Q_n}\big) + \delta_n^k [\alpha_{15} - (k\Omega)^2]\Big) \\
g_{kr2}^{(1c)} &= \sum_{n=1}^{N} \delta_{n+N}^r \Big(\frac{\partial P_n}{\partial c_{1n}}\big(\alpha_2 \frac{\partial f_2^2}{\partial P_n} + \alpha_4 \frac{\partial f_2^4}{\partial P_n} + \alpha_5 \frac{\partial f_2^5}{\partial P_n} + \alpha_8 \frac{\partial f_2^8}{\partial P_n} + \alpha_9 \frac{\partial f_2^9}{\partial P_n} + \\
&\quad \alpha_{10} \frac{\partial f_2^{10}}{\partial P_n}\big) + \delta_n^k k\Omega[\alpha_{11} + \alpha_{14}]\Big) \\
g_{kr3}^{(1c)} &= \sum_{n=1}^{N} \delta_{n+2N+1}^r \Big(\frac{\partial C_n}{\partial b_{2n}}\big(\alpha_1 \frac{\partial f_2^1}{\partial C_n} + \alpha_2 \frac{\partial f_2^2}{\partial C_n} + \alpha_3 \frac{\partial f_2^3}{\partial C_n} + \alpha_4 \frac{\partial f_2^4}{\partial C_n} + \alpha_5 \frac{\partial f_2^5}{\partial C_n} + \\
&\quad \alpha_6 \frac{\partial f_2^6}{\partial C_n}\big) + \alpha_3 \frac{\partial f_2^3}{\partial b_{2n}} + \alpha_5 \frac{\partial f_2^5}{\partial b_{2n}} + \alpha_6 \frac{\partial f_2^6}{\partial b_{2n}} + \alpha_9 \frac{\partial f_2^9}{\partial b_{2n}} + \alpha_{10} \frac{\partial f_2^{10}}{\partial b_{2n}} + \alpha_{12} \frac{\partial f_2^{12}}{\partial b_{2n}} \\
&\quad + \alpha_{13} \frac{\partial f_2^{13}}{\partial b_{2n}}\Big) \\
g_{kr4}^{(1c)} &= \sum_{n=1}^{N} \delta_{n+3N+1}^r \Big(\frac{\partial B_n}{\partial c_{2n}}\big(\alpha_1 \frac{\partial f_2^1}{\partial B_n} + \alpha_2 \frac{\partial f_2^2}{\partial B_n} + \alpha_3 \frac{\partial f_2^3}{\partial B_n} + \alpha_4 \frac{\partial f_2^4}{\partial B_n} + \alpha_5 \frac{\partial f_2^5}{\partial B_n} + \\
&\quad \alpha_6 \frac{\partial f_2^6}{\partial B_n}\big) + \alpha_3 \frac{\partial f_2^3}{\partial c_{2n}} + \alpha_5 \frac{\partial f_2^5}{\partial c_{2n}} + \alpha_6 \frac{\partial f_2^6}{\partial c_{2n}} + \alpha_7 \frac{\partial f_2^7}{\partial c_{2n}} + \alpha_9 \frac{\partial f_2^9}{\partial c_{2n}} + \alpha_{10} \frac{\partial f_2^{10}}{\partial c_{2n}} \\
&\quad + \alpha_{12} \frac{\partial f_2^{12}}{\partial c_{2n}}\Big)
\end{aligned}
\tag{4.32}
$$

The derivative related to the sine term for the transverse motion is

$$
g_{kr}^{(1s)} = g_{kr1}^{(1s)} + g_{kr2}^{(1s)} + g_{kr3}^{(1s)} + g_{kr4}^{(1s)} \tag{4.33}
$$

with

$$
\begin{aligned}
g_{kr1}^{(1s)} &= \sum_{n=1}^{N} \delta_n^r \Big(\frac{\partial Q_n}{\partial b_{1n}}\big(\alpha_2 \frac{\partial f_3^2}{\partial Q_n} + \alpha_4 \frac{\partial f_3^4}{\partial Q_n} + \alpha_5 \frac{\partial f_3^5}{\partial Q_n} + \alpha_8 \frac{\partial f_3^8}{\partial Q_n} + \alpha_9 \frac{\partial f_3^9}{\partial Q_n} + \alpha_{10} \frac{\partial f_3^{10}}{\partial Q_n}\big) \\
&\quad + \delta_n^k [-(\alpha_{11} + \alpha_{14})(k\Omega)]\Big) \\
g_{kr2}^{(1s)} &= \sum_{n=1}^{N} \delta_{n+N}^r \Big(\frac{\partial P_n}{\partial c_{1n}}\big(\alpha_2 \frac{\partial f_3^2}{\partial P_n} + \alpha_4 \frac{\partial f_3^4}{\partial P_n} + \alpha_5 \frac{\partial f_3^5}{\partial P_n} + \alpha_8 \frac{\partial f_3^8}{\partial P_n} + \alpha_9 \frac{\partial f_3^9}{\partial P_n} \alpha_{10} \frac{\partial f_3^{10}}{\partial P_n}\big) \\
&\quad + \delta_n^k [\alpha_{15} - (k\Omega)^2]\Big)
\end{aligned}
$$

$$g_{kr3}^{(1s)}=\sum_{n=1}^{N}\delta_{n+2N+1}^{r}(\frac{\partial C_n}{\partial b_{2n}}(\alpha_1\frac{\partial f_3^1}{\partial C_n}+\alpha_2\frac{\partial f_3^2}{\partial C_n}+\alpha_3\frac{\partial f_3^3}{\partial C_n}+\alpha_4\frac{\partial f_3^4}{\partial C_n}+\alpha_5\frac{\partial f_3^5}{\partial C_n}$$
$$+\alpha_6\frac{\partial f_3^6}{\partial C_n})+\alpha_3\frac{\partial f_3^3}{\partial b_{2n}}+\alpha_5\frac{\partial f_3^5}{\partial b_{2n}}+\alpha_6\frac{\partial f_3^6}{\partial b_{2n}}+\alpha_7\frac{\partial f_3^7}{\partial b_{2n}}+\alpha_9\frac{\partial f_3^9}{\partial b_{2n}}+\alpha_{10}\frac{\partial f_3^{10}}{\partial b_{2n}}+$$
$$\alpha_{12}\frac{\partial f_3^{12}}{\partial b_{2n}})$$
$$g_{kr4}^{(1s)}=\sum_{n=1}^{N}\delta_{n+3N+1}^{r}(\frac{\partial B_n}{\partial c_{2n}}(\alpha_1\frac{\partial f_3^1}{\partial B_n}+\alpha_2\frac{\partial f_3^2}{\partial B_n}+\alpha_3\frac{\partial f_3^3}{\partial B_n}+\alpha_4\frac{\partial f_3^4}{\partial B_n}+\alpha_5\frac{\partial f_3^5}{\partial B_n}$$
$$+\alpha_6\frac{\partial f_3^6}{\partial B_n})+\alpha_3\frac{\partial f_3^3}{\partial c_{2n}}+\alpha_5\frac{\partial f_3^5}{\partial c_{2n}}+\alpha_6\frac{\partial f_3^6}{\partial c_{2n}}+\alpha_9\frac{\partial f_3^9}{\partial c_{2n}}+\alpha_{10}\frac{\partial f_3^{10}}{\partial c_{2n}}+$$
$$\alpha_{12}\frac{\partial f_3^{12}}{\partial c_{2n}}+\alpha_{13}\frac{\partial f_3^{13}}{\partial c_{2n}}) \tag{4.34}$$

The derivative relatives to the constant for the torsional motion are

$$g_r^{(20)}=g_{r1}^{(20)}+g_{r2}^{(20)}+g_{r3}^{(20)}+g_{r4}^{(20)} \tag{4.35}$$

with

$$g_{r1}^{(20)}=\sum_{n=1}^{N}\delta_n^r\frac{\partial Q_n}{\partial b_{1n}}(\beta_2\frac{\partial f_1^2}{\partial Q_n}+\beta_4\frac{\partial f_1^4}{\partial Q_n}+\beta_5\frac{\partial f_1^5}{\partial Q_n}+\beta_8\frac{\partial f_1^8}{\partial Q_n}+\beta_9\frac{\partial f_1^9}{\partial Q_n}$$
$$+\beta_{10}\frac{\partial f_1^{10}}{\partial Q_n})$$
$$g_{r2}^{(20)}=\sum_{n=1}^{N}\delta_{n+N}^r\frac{\partial P_n}{\partial c_{1n}}(\beta_2\frac{\partial f_1^2}{\partial P_n}+\beta_4\frac{\partial f_1^4}{\partial P_n}+\beta_5\frac{\partial f_1^5}{\partial P_n}+\beta_8\frac{\partial f_1^8}{\partial P_n}+\beta_9\frac{\partial f_1^9}{\partial P_n}$$
$$+\beta_{10}\frac{\partial f_1^{10}}{\partial P_n}) \tag{4.36}$$
$$g_{r3}^{(20)}=\sum_{n=1}^{N}\delta_{n+2N+1}^r(\frac{\partial C_n}{\partial b_{2n}}(\beta_1\frac{\partial f_1^1}{\partial C_n}+\beta_2\frac{\partial f_1^2}{\partial C_n}+\beta_3\frac{\partial f_1^3}{\partial C_n}+\beta_4\frac{\partial f_1^4}{\partial C_n}+\beta_5\frac{\partial f_1^5}{\partial C_n}$$
$$+\beta_6\frac{\partial f_1^6}{\partial C_n})+\beta_3\frac{\partial f_1^3}{\partial b_{2n}}+\beta_5\frac{\partial f_1^5}{\partial b_{2n}}+\beta_6\frac{\partial f_1^6}{\partial b_{2n}}+\beta_9\frac{\partial f_1^9}{\partial b_{2n}}+\beta_{10}\frac{\partial f_1^{10}}{\partial b_{2n}}$$
$$+\beta_{12}\frac{\partial f_1^{12}}{\partial b_{2n}})$$

$$g_{r4}^{(20)} = \sum_{n=1}^{N} \delta_{n+3N+1}^{r} (\frac{\partial B_n}{\partial c_{2n}} (\beta_1 \frac{\partial f_1^1}{\partial B_n} + \beta_2 \frac{\partial f_1^2}{\partial B_n} + \beta_3 \frac{\partial f_1^3}{\partial B_n} + \beta_4 \frac{\partial f_1^4}{\partial B_n} + \beta_5 \frac{\partial f_1^5}{\partial B_n}$$
$$+ \beta_6 \frac{\partial f_1^6}{\partial B_n}) + \beta_3 \frac{\partial f_1^3}{\partial c_{2n}} + \beta_5 \frac{\partial f_1^5}{\partial c_{2n}} + \beta_6 \frac{\partial f_1^6}{\partial c_{2n}} + \beta_9 \frac{\partial f_1^9}{\partial c_{2n}} + \beta_{10} \frac{\partial f_1^{10}}{\partial c_{2n}}$$
$$+ \beta_{12} \frac{\partial f_1^{12}}{\partial c_{2n}})$$

The derivative related to the cosine term for the torsional motion is

$$g_{kr}^{(2c)} = g_{kr1}^{(2c)} + g_{kr2}^{(2c)} + g_{kr3}^{(2c)} + g_{kr4}^{(2c)} \tag{4.37}$$

where

$$g_{kr1}^{(\partial c)} = \sum_{n=1}^{N} \delta_n^r \frac{\partial Q_n}{\partial b_{1n}} (\beta_2 \frac{\partial f_2^2}{\partial Q_n} + \beta_4 \frac{\partial f_2^4}{\partial Q_n} + \beta_5 \frac{\partial f_2^5}{\partial Q_n} + \beta_8 \frac{\partial f_2^8}{\partial Q_n} + \beta_9 \frac{\partial f_2^9}{\partial Q_n}$$
$$+ \beta_{10} \frac{\partial f_2^{10}}{\partial Q_n})$$
$$g_{kr2}^{(2c)} = \sum_{n=1}^{N} \delta_{n+N}^r (\frac{\partial P_i}{\partial c_{1n}} (\beta_2 \frac{\partial f_2^2}{\partial P_n} + \beta_4 \frac{\partial f_2^4}{\partial P_n} + \beta_5 \frac{\partial f_2^5}{\partial P_n} + \beta_8 \frac{\partial f_2^8}{\partial P_n} + \beta_9 \frac{\partial f_2^9}{\partial P_n}$$
$$+ \beta_{10} \frac{\partial f_2^{10}}{\partial P_n}) + \delta_n^k \beta_{11}) \tag{4.38}$$
$$g_{kr3}^{(2c)} = \sum_{n=1}^{N} \delta_{n+2N+1}^r (\frac{\partial C_n}{\partial b_{2n}} (\beta_1 \frac{\partial f_2^1}{\partial C_n} + \beta_2 \frac{\partial f_2^2}{\partial C_n} + \beta_3 \frac{\partial f_2^3}{\partial C_n} + \beta_4 \frac{\partial f_2^4}{\partial C_n} + \beta_5 \frac{\partial f_2^5}{\partial C_n}$$
$$+ \beta_6 \frac{\partial f_2^6}{\partial C_n}) + \beta_3 \frac{\partial f_2^3}{\partial b_{2n}} + \beta_5 \frac{\partial f_2^5}{\partial b_{2n}} + \beta_6 \frac{\partial f_2^6}{\partial b_{2n}} + \beta_9 \frac{\partial f_2^9}{\partial b_{2n}} + \beta_{10} \frac{\partial f_2^{10}}{\partial b_{2n}} +$$
$$\beta_{12} \frac{\partial f_2^{12}}{\partial b_{2n}} + \beta_{13} \frac{\partial f_2^{13}}{\partial b_{2n}} + \beta_{15} \frac{\partial f_2^{15}}{\partial b_{2n}} - \delta_n^k (k\Omega)^2)$$
$$g_{kr4}^{(2c)} = \sum_{n=1}^{N} \delta_{n+3N+1}^r (\frac{\partial B_n}{\partial c_{2n}} (\beta_1 \frac{\partial f_2^1}{\partial B_n} + \beta_2 \frac{\partial f_2^2}{\partial B_n} + \beta_3 \frac{\partial f_2^3}{\partial B_n} + \beta_4 \frac{\partial f_2^4}{\partial B_n} + \beta_5 \frac{\partial f_2^5}{\partial B_n}$$
$$+ \beta_6 \frac{\partial f_2^6}{\partial B_n}) + \beta_3 \frac{\partial f_2^3}{\partial c_{2n}} + \beta_5 \frac{\partial f_2^5}{\partial c_{2n}} + \beta_6 \frac{\partial f_2^6}{\partial c_{2n}} + \beta_9 \frac{\partial f_2^9}{\partial c_{2n}} + \beta_{10} \frac{\partial f_2^{10}}{\partial c_{2n}} +$$
$$\beta_{12} \frac{\partial f_2^{12}}{\partial c_{2n}} + \beta_{12} \frac{\partial f_2^{14}}{\partial c_{2n}})$$

The derivative related to the sine term for the torsional motion is

$$g_{kr}^{(2s)} = g_{kr1}^{(2s)} + g_{kr2}^{(2s)} + g_{kr3}^{(2s)} + g_{kr4}^{(2s)} \tag{4.39}$$

where

$$\begin{aligned}
g_{kr1}^{(2s)} &= \sum_{n=1}^{N} \delta_n^r \Big(\frac{\partial Q_n}{\partial b_{1n}}\Big(\beta_2 \frac{\partial f_3^2}{\partial Q_n} + \beta_4 \frac{\partial f_3^4}{\partial Q_n} + \beta_5 \frac{\partial f_3^5}{\partial Q_n} + \beta_8 \frac{\partial f_3^8}{\partial Q_n} + \beta_9 \frac{\partial f_3^9}{\partial Q_n} \\
&\quad + \beta_{10} \frac{\partial f_3^{10}}{\partial Q_n}\Big) - \delta_n^k \beta_{11} k\Omega\Big) \\
g_{kr2}^{(2s)} &= \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial P_n}{\partial c_{1n}}\Big(\beta_2 \frac{\partial f_3^2}{\partial P_n} + \beta_4 \frac{\partial f_3^4}{\partial P_n} + \beta_5 \frac{\partial f_3^5}{\partial P_n} + \beta_8 \frac{\partial f_3^8}{\partial P_n} + \beta_9 \frac{\partial f_3^9}{\partial P_n} \\
&\quad + \beta_{10} \frac{\partial f_3^{10}}{\partial P_n}\Big) \\
g_{kr3}^{(2s)} &= \sum_{n=1}^{N} \delta_{n+2N+1}^r \Big(\frac{\partial C_n}{\partial b_{2n}}\Big(\beta_1 \frac{\partial f_3^1}{\partial C_n} + \beta_2 \frac{\partial f_3^2}{\partial C_n} + \beta_3 \frac{\partial f_3^3}{\partial C_n} + \beta_4 \frac{\partial f_3^4}{\partial C_n} + \beta_5 \frac{\partial f_3^5}{\partial C_n} \\
&\quad + \beta_6 \frac{\partial f_3^6}{\partial C_n}\Big) + \beta_3 \frac{\partial f_3^3}{\partial b_{2n}} + \beta_5 \frac{\partial f_3^5}{\partial b_{2n}} + \beta_6 \frac{\partial f_3^6}{\partial b_{2n}} + \beta_7 \frac{\partial f_3^7}{\partial b_{2n}} + \beta_9 \frac{\partial f_3^9}{\partial b_{2n}} + \\
&\quad \beta_{10} \frac{\partial f_3^{10}}{\partial b_{2n}} + \beta_{12} \frac{\partial f_3^{12}}{\partial b_{2n}} + \beta_{14} \frac{\partial f_3^{14}}{\partial b_{2n}} \\
g_{kr4}^{(2s)} &= \sum_{n=1}^{N} \delta_{n+3N+1}^r \Big(\frac{\partial B_n}{\partial c_{2n}}\Big(\beta_1 \frac{\partial f_3^1}{\partial B_n} + \beta_2 \frac{\partial f_3^2}{\partial B_n} + \beta_3 \frac{\partial f_3^3}{\partial B_n} + \beta_4 \frac{\partial f_3^4}{\partial B_n} + \beta_5 \frac{\partial f_3^5}{\partial B_n} \\
&\quad + \beta_6 \frac{\partial f_3^6}{\partial B_n}\Big) + \beta_3 \frac{\partial f_3^3}{\partial c_{2n}} + \beta_5 \frac{\partial f_3^5}{\partial c_{2n}} + \beta_6 \frac{\partial f_3^6}{\partial c_{2n}} + \beta_9 \frac{\partial f_3^9}{\partial c_{2n}} + \beta_{10} \frac{\partial f_3^{10}}{\partial c_{2n}} + \\
&\quad \beta_{12} \frac{\partial f_3^{12}}{\partial c_{2n}} + \beta_{13} \frac{\partial f_3^{13}}{\partial c_{2n}} + \beta_{15} \frac{\partial f_3^{15}}{\partial c_{2n}} - \delta_n^k (k\Omega)^2\Big)
\end{aligned} \tag{4.40}$$

The H-matrix is

$$\mathbf{H} = \frac{\partial \mathbf{g}^{(m)}}{\partial \mathbf{z}_1^{(m)}} = (\mathbf{H}^{(10)}, \mathbf{H}^{(1c)}, \mathbf{H}^{(1s)}, \mathbf{H}^{(20)}, \mathbf{H}^{(2c)}, \mathbf{H}^{(2s)})^{\mathrm{T}} \tag{4.41}$$

where

$$\begin{aligned}
\mathbf{H}^{(i0)} &= (H_0^{(i0)}, H_1^{(i0)}, \cdots, H_{4N+1}^{(i0)}), \\
\mathbf{H}^{(ic)} &= (\mathbf{H}_1^{(ic)}, \mathbf{H}_2^{(ic)}, \cdots, \mathbf{H}_N^{(ic)})^{\mathrm{T}}, \\
\mathbf{H}^{(is)} &= (\mathbf{H}_1^{(is)}, \mathbf{H}_2^{(is)}, \cdots, \mathbf{H}_N^{(is)})^{\mathrm{T}}
\end{aligned} \tag{4.42}$$

for $i = 1, 2$ and $N = 1, 2, \cdots \infty$, with

$$\begin{aligned}
\mathbf{H}_k^{(ic)} &= (H_{k0}^{(ic)}, H_{k1}^{(ic)}, \cdots, H_{k(4N+1)}^{(ic)}), \\
\mathbf{H}_k^{(is)} &= (H_{k0}^{(is)}, H_{k1}^{(is)}, \cdots, H_{k(4N+1)}^{(is)})
\end{aligned} \tag{4.43}$$

for $k = 1, 2, \cdots N$. The corresponding components are

$$\begin{aligned}
H_r^{(10)} = & -\delta_0^r (\alpha_2 \frac{\partial f_1^2}{\partial \dot{a}_{10}} + \alpha_4 \frac{\partial f_1^4}{\partial \dot{a}_{10}} + \alpha_5 \frac{\partial f_1^5}{\partial \dot{a}_{10}} + \alpha_8 \frac{\partial f_1^8}{\partial \dot{a}_{10}} + \alpha_9 \frac{\partial f_1^9}{\partial \dot{a}_{10}} + \alpha_{10} \frac{\partial f_1^{10}}{\partial \dot{a}_{10}} \\
& + \alpha_{11} + \alpha_{14}) - \delta_{2N+1}^r (\alpha_1 \frac{\partial f_1^1}{\partial \dot{a}_{20}} + \alpha_2 \frac{\partial f_1^2}{\partial \dot{a}_{20}} + \alpha_3 \frac{\partial f_1^3}{\partial \dot{a}_{20}} + \alpha_4 \frac{\partial f_1^4}{\partial \dot{a}_{20}} + \alpha_5 \frac{\partial f_1^5}{\partial \dot{a}_{20}} \\
& + \alpha_6 \frac{\partial f_1^6}{\partial \dot{a}_{20}} + \alpha_7) - Z_r^{(10)}, \\
H_{kr}^{(1c)} = & -\delta_0^r (\alpha_2 \frac{\partial f_2^2}{\partial \dot{a}_{10}} + \alpha_4 \frac{\partial f_2^4}{\partial \dot{a}_{10}} + \alpha_5 \frac{\partial f_2^5}{\partial \dot{a}_{10}} + \alpha_8 \frac{\partial f_2^8}{\partial \dot{a}_{10}} + \alpha_9 \frac{\partial f_2^9}{\partial \dot{a}_{10}} + \alpha_{10} \frac{\partial f_2^{10}}{\partial \dot{a}_{10}}) \\
& - \delta_{2N+1}^r (\alpha_1 \frac{\partial f_2^1}{\partial \dot{a}_{20}} + \alpha_2 \frac{\partial f_2^2}{\partial \dot{a}_{20}} + \alpha_3 \frac{\partial f_2^3}{\partial \dot{a}_{20}} + \alpha_4 \frac{\partial f_2^4}{\partial \dot{a}_{20}} + \alpha_5 \frac{\partial f_2^5}{\partial \dot{a}_{20}} + \alpha_6 \frac{\partial f_2^6}{\partial \dot{a}_{20}} \\
& + \alpha_9 \frac{\partial f_2^9}{\partial \dot{a}_{20}}) - Z_{kr}^{(1c)}, \\
H_{kr}^{(1s)} = & -\delta_0^r (\alpha_2 \frac{\partial f_3^2}{\partial \dot{a}_{10}} + \alpha_4 \frac{\partial f_3^4}{\partial \dot{a}_{10}} + \alpha_5 \frac{\partial f_3^5}{\partial \dot{a}_{10}} + \alpha_8 \frac{\partial f_3^8}{\partial \dot{a}_{10}} + \alpha_9 \frac{\partial f_3^9}{\partial \dot{a}_{10}} + \alpha_{10} \frac{\partial f_3^{10}}{\partial \dot{a}_{10}}) \\
& - \delta_{2N+1}^r (\alpha_1 \frac{\partial f_3^1}{\partial \dot{a}_{20}} + \alpha_2 \frac{\partial f_3^2}{\partial \dot{a}_{20}} + \alpha_3 \frac{\partial f_3^3}{\partial \dot{a}_{20}} + \alpha_4 \frac{\partial f_3^4}{\partial \dot{a}_{20}} + \alpha_5 \frac{\partial f_3^5}{\partial \dot{a}_{20}} + \alpha_6 \frac{\partial f_3^6}{\partial \dot{a}_{20}}) \\
& - Z_{kr}^{(1s)},
\end{aligned} \tag{4.44}$$

$$
\begin{aligned}
H_r^{(20)} &= -\delta_0^r(\beta_2 \frac{\partial f_1^2}{\partial \dot{a}_{10}} + \beta_4 \frac{\partial f_1^4}{\partial \dot{a}_{10}} + \beta_5 \frac{\partial f_1^5}{\partial \dot{a}_{10}} + \beta_8 \frac{\partial f_1^8}{\partial \dot{a}_{10}} + \beta_9 \frac{\partial f_1^9}{\partial \dot{a}_{10}} + \beta_{10} \frac{\partial f_1^{10}}{\partial \dot{a}_{10}} \\
&\quad + \alpha_{11}) - \delta_{2N+1}^r(\beta_1 \frac{\partial f_1^1}{\partial \dot{a}_{20}} + \beta_2 \frac{\partial f_1^2}{\partial \dot{a}_{20}} + \beta_3 \frac{\partial f_1^3}{\partial \dot{a}_{20}} + \beta_4 \frac{\partial f_1^4}{\partial \dot{a}_{20}} + \beta_5 \frac{\partial f_1^5}{\partial \dot{a}_{20}} \\
&\quad + \beta_6 \frac{\partial f_1^6}{\partial \dot{a}_{20}} + \beta_7 + \beta_{14}) - Z_r^{(20)}, \\
H_{kr}^{(2c)} &= -\delta_0^r(\beta_2 \frac{\partial f_2^2}{\partial \dot{a}_{10}} + \beta_4 \frac{\partial f_2^4}{\partial \dot{a}_{10}} + \beta_5 \frac{\partial f_2^5}{\partial \dot{a}_{10}} + \beta_8 \frac{\partial f_2^8}{\partial \dot{a}_{10}} + \beta_9 \frac{\partial f_2^9}{\partial \dot{a}_{10}} + \beta_{10} \frac{\partial f_2^{10}}{\partial \dot{a}_{10}}) \\
&\quad - \delta_{2N+1}^r(\beta_1 \frac{\partial f_2^1}{\partial \dot{a}_{20}} + \beta_2 \frac{\partial f_2^2}{\partial \dot{a}_{20}} + \beta_3 \frac{\partial f_2^3}{\partial \dot{a}_{20}} + \beta_4 \frac{\partial f_2^4}{\partial \dot{a}_{20}} + \beta_5 \frac{\partial f_2^5}{\partial \dot{a}_{20}} + \beta_6 \frac{\partial f_2^6}{\partial \dot{a}_{20}} \\
&\quad + \beta_9 \frac{\partial f_2^9}{\partial \dot{a}_{20}}) - Z_{kr}^{(2c)}, \\
H_{kr}^{(2s)} &= -\delta_0^r(\beta_2 \frac{\partial f_3^2}{\partial \dot{a}_{10}} + \beta_4 \frac{\partial f_3^4}{\partial \dot{a}_{10}} + \beta_5 \frac{\partial f_3^5}{\partial \dot{a}_{10}} + \beta_8 \frac{\partial f_3^8}{\partial \dot{a}_{10}} + \beta_9 \frac{\partial f_3^9}{\partial \dot{a}_{10}} + \beta_{10} \frac{\partial f_3^{10}}{\partial \dot{a}_{10}}) \\
&\quad - \delta_{2N+1}^r(\beta_1 \frac{\partial f_3^1}{\partial \dot{a}_{20}} + \beta_2 \frac{\partial f_3^2}{\partial \dot{a}_{20}} + \beta_3 \frac{\partial f_3^3}{\partial \dot{a}_{20}} + \beta_4 \frac{\partial f_3^4}{\partial \dot{a}_{20}} + \beta_5 \frac{\partial f_3^5}{\partial \dot{a}_{20}} + \beta_6 \frac{\partial f_3^6}{\partial \dot{a}_{20}}) \\
&\quad - Z_{kr}^{(2s)}
\end{aligned} \tag{4.45}
$$

for $r = 0, 1, \cdots, 4N+1$.

The derivative of the constant term for the transverse motion is

$$
Z_r^{(10)} = Z_{r1}^{(10)} + Z_{r2}^{(10)} + Z_{r3}^{(10)} + Z_{r4}^{(10)}, \tag{4.46}
$$

with

$$
\begin{aligned}
Z_{r1}^{(10)} &= \sum_{n=1}^{N} \delta_n^r \frac{\partial P_n}{\partial \dot{b}_{1n}} (\alpha_2 \frac{\partial f_1^2}{\partial P_n} + \alpha_4 \frac{\partial f_1^4}{\partial P_n} + \alpha_5 \frac{\partial f_1^5}{\partial P_n} + \alpha_8 \frac{\partial f_1^8}{\partial P_n} \\
&\quad + \alpha_9 \frac{\partial f_1^9}{\partial P_n} + \alpha_{10} \frac{\partial f_1^{10}}{\partial P_n}) \\
Z_{r2}^{(10)} &= \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial Q_n}{\partial \dot{c}_{1n}} (\alpha_2 \frac{\partial f_1^2}{\partial Q_n} + \alpha_4 \frac{\partial f_1^4}{\partial Q_n} + \alpha_5 \frac{\partial f_1^5}{\partial Q_n} + \alpha_8 \frac{\partial f_1^8}{\partial Q_n} \\
&\quad + \alpha_9 \frac{\partial f_1^9}{\partial Q_n} + \alpha_{10} \frac{\partial f_1^{10}}{\partial Q_n})
\end{aligned} \tag{4.47}
$$

$$Z_{r3}^{(10)} = \sum_{n=1}^{N} \delta_{n+2N+1}^{r} \frac{\partial B_n}{\partial \dot{b}_{2n}} (\alpha_1 \frac{\partial f_1^1}{\partial B_n} + \alpha_2 \frac{\partial f_1^2}{\partial B_n} + \alpha_3 \frac{\partial f_1^3}{\partial B_n} + \alpha_4 \frac{\partial f_1^4}{\partial B_n}$$
$$+\alpha_5 \frac{\partial f_1^5}{\partial B_n} + \alpha_6 \frac{\partial f_1^6}{\partial B_n})$$
$$Z_{r4}^{(10)} = \sum_{n=1}^{N} \delta_{n+3N+1}^{r} \frac{\partial C_n}{\partial \dot{c}_{2n}} (\alpha_1 \frac{\partial f_1^1}{\partial C_n} + \alpha_2 \frac{\partial f_1^2}{\partial C_n} + \alpha_3 \frac{\partial f_1^3}{\partial C_n} + \alpha_4 \frac{\partial f_1^4}{\partial C_n}$$
$$+\alpha_5 \frac{\partial f_1^5}{\partial C_n} + \alpha_6 \frac{\partial f_1^6}{\partial C_n})$$

The derivative of the constant term for the torsional motion is

$$Z_r^{(20)} = Z_{r1}^{(20)} + Z_{r2}^{(20)} + Z_{r3}^{(20)} + Z_{r4}^{(20)}, \tag{4.48}$$

with

$$Z_{r1}^{(20)} = \sum_{n=1}^{N} \delta_n^r \frac{\partial P_n}{\partial \dot{b}_{1n}} (\beta_2 \frac{\partial f_1^2}{\partial P_n} + \beta_4 \frac{\partial f_1^4}{\partial P_n} + \beta_5 \frac{\partial f_1^5}{\partial P_n} + \beta_8 \frac{\partial f_1^8}{\partial P_n}$$
$$+\beta_9 \frac{\partial f_1^9}{\partial P_n} + \beta_{10} \frac{\partial f_1^{10}}{\partial P_n})$$
$$Z_{r2}^{(20)} = \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial Q_n}{\partial \dot{c}_{1n}} (\beta_2 \frac{\partial f_1^2}{\partial Q_n} + \beta_4 \frac{\partial f_1^4}{\partial Q_i} + \beta_5 \frac{\partial f_1^5}{\partial Q_i} + \beta_8 \frac{\partial f_1^8}{\partial Q_i}$$
$$+\beta_9 \frac{\partial f_1^9}{\partial Q_i} + \beta_{10} \frac{\partial f_1^{10}}{\partial Q_i}) \tag{4.49}$$
$$Z_{r3}^{(20)} = \sum_{n=1}^{N} \delta_{n+2N+1}^r \frac{\partial B_n}{\partial \dot{b}_{2n}} (\beta_1 \frac{\partial f_1^1}{\partial B_n} + \beta_2 \frac{\partial f_1^2}{\partial B_n} + \beta_3 \frac{\partial f_1^3}{\partial B_n} + \beta_4 \frac{\partial f_1^4}{\partial B_n}$$
$$+\beta_5 \frac{\partial f_1^5}{\partial B_n} + \beta_6 \frac{\partial f_1^6}{\partial B_n})$$
$$Z_{r4}^{(20)} = \sum_{n=1}^{N} \delta_{n+3N+1}^r \frac{\partial C_n}{\partial \dot{c}_{2n}} (\beta_1 \frac{\partial f_1^1}{\partial C_n} + \beta_2 \frac{\partial f_1^2}{\partial C_n} + \beta_3 \frac{\partial f_1^3}{\partial C_n} + \beta_4 \frac{\partial f_1^4}{\partial C_n}$$
$$+\beta_5 \frac{\partial f_1^5}{\partial C_n} + \beta_6 \frac{\partial f_1^6}{\partial C_n})$$

The derivative of the cosine term for the transverse motion is

$$Z_{kr}^{(1c)} = Z_{kr1}^{(1c)} + Z_{kr2}^{(1c)} + Z_{kr3}^{(1c)} + Z_{kr4}^{(1c)} \tag{4.50}$$

with

$$Z_{kr1}^{(1c)} = \sum_{n=1}^{N} \delta_n^r (\frac{\partial P_n}{\partial \dot{b}_{1n}} (\alpha_2 \frac{\partial f_2^2}{\partial P_n} + \alpha_4 \frac{\partial f_2^4}{\partial P_n} + \alpha_5 \frac{\partial f_2^5}{\partial P_n} + \alpha_8 \frac{\partial f_2^8}{\partial P_n} + \alpha_9 \frac{\partial f_2^9}{\partial P_n}$$
$$+\alpha_{10} \frac{\partial f_2^{10}}{\partial P_n}) - \delta_n^k [\alpha_{11} + \alpha_{14}])$$
$$Z_{kr2}^{(1c)} = \sum_{n=1}^{N} \delta_{n+N}^r (\frac{\partial Q_n}{\partial \dot{c}_{1n}} (\alpha_2 \frac{\partial f_2^2}{\partial Q_n} + \alpha_4 \frac{\partial f_2^4}{\partial Q_n} + \alpha_5 \frac{\partial f_2^5}{\partial Q_n} + \alpha_8 \frac{\partial f_2^8}{\partial Q_n} + \alpha_9 \frac{\partial f_2^9}{\partial Q_n}$$
$$+\alpha_{10} \frac{\partial f_2^{10}}{\partial Q_n}) - \delta_n^k 2k\Omega) \tag{4.51}$$
$$Z_{kr3}^{(1c)} = \sum_{n=1}^{N} \delta_{n+2N+1}^r (\frac{\partial B_n}{\partial \dot{b}_{2n}} (\alpha_1 \frac{\partial f_2^1}{\partial B_n} + \alpha_2 \frac{\partial f_2^2}{\partial B_n} + \alpha_3 \frac{\partial f_2^3}{\partial B_n} + \alpha_4 \frac{\partial f_2^4}{\partial B_n} + \alpha_5 \frac{\partial f_2^5}{\partial B_n}$$
$$+\alpha_6 \frac{\partial f_2^6}{\partial B_n}) - \delta_n^k \alpha_7)$$
$$Z_{kr4}^{(1c)} = \sum_{n=1}^{N} \delta_{n+3N+1}^r \frac{\partial C_n}{\partial \dot{c}_{2n}} (\alpha_1 \frac{\partial f_2^1}{\partial C_n} + \alpha_2 \frac{\partial f_2^2}{\partial C_n} + \alpha_3 \frac{\partial f_2^3}{\partial C_n} + \alpha_4 \frac{\partial f_2^4}{\partial C_n} + \alpha_5 \frac{\partial f_2^5}{\partial C_n}$$
$$+\alpha_6 \frac{\partial f_2^6}{\partial C_n})$$

The derivative of the cosine term for the torsional motion is

$$Z_{kr}^{(2c)} = Z_{kr1}^{(2c)} + Z_{kr2}^{(2c)} + Z_{kr3}^{(2c)} + Z_{kr4}^{(2c)} \tag{4.52}$$

with

$$Z_{kr1}^{(2c)} = \sum_{n=1}^{N} \delta_n^r (\frac{\partial P_i}{\partial \dot{b}_{1n}} (\beta_2 \frac{\partial f_2^2}{\partial P_n} + \beta_4 \frac{\partial f_2^4}{\partial P_n} + \beta_5 \frac{\partial f_2^5}{\partial P_n} + \beta_8 \frac{\partial f_2^8}{\partial P_n} + \beta_9 \frac{\partial f_2^9}{\partial P_n}$$
$$+\beta_{10} \frac{\partial f_2^{10}}{\partial P_n}) - \delta_n^k \beta_{11})$$
$$Z_{kr2}^{(2c)} = \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial Q_n}{\partial \dot{c}_{1n}} (\beta_2 \frac{\partial f_2^2}{\partial Q_n} + \beta_4 \frac{\partial f_2^4}{\partial Q_n} + \beta_5 \frac{\partial f_2^5}{\partial Q_n} + \beta_8 \frac{\partial f_2^8}{\partial Q_n} + \beta_9 \frac{\partial f_2^9}{\partial Q_n}$$
$$+\beta_{10} \frac{\partial f_2^{10}}{\partial Q_n}) \tag{4.53}$$
$$Z_{kr3}^{(2c)} = \sum_{n=1}^{N} \delta_{n+2N+1}^r (\frac{\partial B_n}{\partial \dot{b}_{2n}} (\beta_1 \frac{\partial f_2^1}{\partial B_n} + \beta_2 \frac{\partial f_2^2}{\partial B_n} + \beta_3 \frac{\partial f_2^3}{\partial B_n} + \beta_4 \frac{\partial f_2^4}{\partial B_n} + \beta_5 \frac{\partial f_2^5}{\partial B_n}$$
$$+\beta_6 \frac{\partial f_2^6}{\partial B_n}) - \delta_n^k [\beta_7 + \beta_{14}])$$

$$Z_{kr4}^{(2c)} = \sum_{n=1}^{N} \delta_{n+3N+1}^{r} (\frac{\partial C_n}{\partial \dot{c}_{2n}} (\beta_1 \frac{\partial f_2^1}{\partial C_n} + \beta_2 \frac{\partial f_2^2}{\partial C_n} + \beta_3 \frac{\partial f_2^3}{\partial C_n} + \beta_4 \frac{\partial f_2^4}{\partial C_n} + \beta_5 \frac{\partial f_2^5}{\partial C_n} + \beta_6 \frac{\partial f_2^6}{\partial C_n}) - \delta_n^k [2k\Omega])$$

The derivative of the sine term for the transverse motion is

$$Z_{kr}^{(1s)} = Z_{kr1}^{(1s)} + Z_{kr2}^{(1s)} + Z_{kr3}^{(1s)} + Z_{kr4}^{(1s)} \tag{4.54}$$

with

$$Z_{kr1}^{(1s)} = \sum_{n=1}^{N} \delta_n^r (\frac{\partial P_n}{\partial \dot{b}_{1n}} (\alpha_2 \frac{\partial f_3^2}{\partial P_n} + \alpha_4 \frac{\partial f_3^4}{\partial P_n} + \alpha_5 \frac{\partial f_3^5}{\partial P_n} + \alpha_8 \frac{\partial f_3^8}{\partial P_n} + \alpha_9 \frac{\partial f_3^9}{\partial P_n} + \alpha_{10} \frac{\partial f_3^{10}}{\partial P_n}) + \delta_n^k 2k\Omega)$$

$$Z_{kr2}^{(1s)} = \sum_{n=1}^{N} \delta_{n+N}^r (\frac{\partial Q_n}{\partial \dot{c}_{1n}} (\alpha_2 \frac{\partial f_3^2}{\partial Q_n} + \alpha_4 \frac{\partial f_3^4}{\partial Q_n} + \alpha_5 \frac{\partial f_3^5}{\partial Q_n} + \alpha_8 \frac{\partial f_3^8}{\partial Q_n} + \alpha_9 \frac{\partial f_3^9}{\partial Q_n} + \alpha_{10} \frac{\partial f_3^{10}}{\partial Q_n}) - \delta_n^k [\alpha_{11} + \alpha_{14}])$$

$$Z_{kr3}^{(1s)} = \sum_{n=1}^{N} \delta_{n+2N+1}^r \frac{\partial B_n}{\partial \dot{b}_{2n}} (\alpha_1 \frac{\partial f_3^1}{\partial B_n} + \alpha_2 \frac{\partial f_3^2}{\partial B_n} + \alpha_3 \frac{\partial f_3^3}{\partial B_n} + \alpha_4 \frac{\partial f_3^4}{\partial B_n} + \alpha_5 \frac{\partial f_3^5}{\partial B_n} + \alpha_6 \frac{\partial f_3^6}{\partial B_n}) \tag{4.55}$$

$$Z_{kr4}^{(1s)} = \sum_{n=1}^{N} \delta_{n+3N+1}^r (\frac{\partial C_n}{\partial \dot{c}_{2n}} (\alpha_1 \frac{\partial f_3^1}{\partial C_n} + \alpha_2 \frac{\partial f_3^2}{\partial C_n} + \alpha_3 \frac{\partial f_3^3}{\partial C_n} + \alpha_4 \frac{\partial f_3^4}{\partial C_n} + \alpha_5 \frac{\partial f_3^5}{\partial C_n} + \alpha_6 \frac{\partial f_3^6}{\partial C_n}) - \delta_n^k \alpha_7)$$

The derivative of the sine term for the torsional motion is

$$Z_{kr}^{(2s)} = Z_{kr1}^{(2s)} + Z_{kr2}^{(2s)} + Z_{kr3}^{(2s)} + Z_{kr4}^{(2s)} \tag{4.56}$$

with

$$Z_{kr1}^{(2s)} = \sum_{n=1}^{N} \delta_n^r \frac{\partial P_n}{\partial \dot{b}_{1n}} (\beta_2 \frac{\partial f_3^2}{\partial P_n} + \beta_4 \frac{\partial f_3^4}{\partial P_n} + \beta_5 \frac{\partial f_3^5}{\partial P_n} + \beta_8 \frac{\partial f_3^8}{\partial P_n} + \beta_9 \frac{\partial f_3^9}{\partial P_n} +$$
$$\beta_{10} \frac{\partial f_3^{10}}{\partial P_n})$$
$$Z_{kr2}^{(2s)} = \sum_{n=1}^{N} \delta_{n+N}^r (\frac{\partial Q_n}{\partial \dot{c}_{1n}} (\beta_2 \frac{\partial f_3^2}{\partial Q_n} + \beta_4 \frac{\partial f_3^4}{\partial Q_n} + \beta_5 \frac{\partial f_3^5}{\partial Q_n} + \beta_8 \frac{\partial f_3^8}{\partial Q_n} + \beta_9 \frac{\partial f_3^9}{\partial Q_n}$$
$$+\beta_{10} \frac{\partial f_3^{10}}{\partial Q_n}) - \delta_n^k \beta_{11}) \tag{4.57}$$
$$Z_{kr3}^{(2s)} = \sum_{n=1}^{N} \delta_{n+2N+1}^r (\frac{\partial B_n}{\partial \dot{b}_{2n}} (\beta_1 \frac{\partial f_3^1}{\partial B_n} + \beta_2 \frac{\partial f_3^2}{\partial B_n} + \beta_3 \frac{\partial f_3^3}{\partial B_n} + \beta_4 \frac{\partial f_3^4}{\partial B_n} + \beta_5 \frac{\partial f_3^5}{\partial B_n}$$
$$+\beta_6 \frac{\partial f_3^6}{\partial B_n}) + \delta_n^k 2k\Omega)$$
$$Z_{kr4}^{(2s)} = \sum_{n=1}^{N} (\delta_{n+3N+1}^r \frac{\partial C_n}{\partial \dot{c}_{2n}} (\beta_1 \frac{\partial f_3^1}{\partial C_n} + \beta_2 \frac{\partial f_3^2}{\partial C_n} + \beta_3 \frac{\partial f_3^3}{\partial C_n} + \beta_4 \frac{\partial f_3^4}{\partial C_n} + \beta_5 \frac{\partial f_3^5}{\partial C_n} +$$
$$\beta_6 \frac{\partial f_3^6}{\partial C_n}) - \delta_n^k [\beta_7 + \beta_{14}])$$

From Luo [2012], the eigenvalues of $D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)})$ are classified as

$$(n_1, n_2, n_3 \mid n_4, n_5, n_6) \tag{4.58}$$

The corresponding boundary between the stable and unstable solutions is given by the saddle-node bifurcation and Hopf bifurcation.

4.2 Frequency-Amplitude Characteristics

The curves of harmonic amplitude varying with excitation frequency Ω are illustrated. The corresponding solution in Eq.(4.7) can be re-written as

$$\begin{aligned} v^*(t) &= a_{10}^{(m)} + \sum_{k=1}^{N} A_{(1)k/m} \cos(\frac{k}{m}\Omega t - \varphi_{(1)k/m}), \\ \theta^*(t) &= a_{20}^{(m)} + \sum_{k=1}^{N} A_{(2)k/m} \cos(\frac{k}{m}\Omega t - \varphi_{(2)k/m}), \end{aligned} \tag{4.59}$$

where the harmonic amplitude and phase are defined by

$$A_{(i)k/m} \equiv \sqrt{b_{ik/m}^2 + c_{ik/m}^2},\ \varphi_{(i)k/m} = \arctan(c_{ik/m} / b_{ik/m}) \tag{4.60}$$

The system parameters are

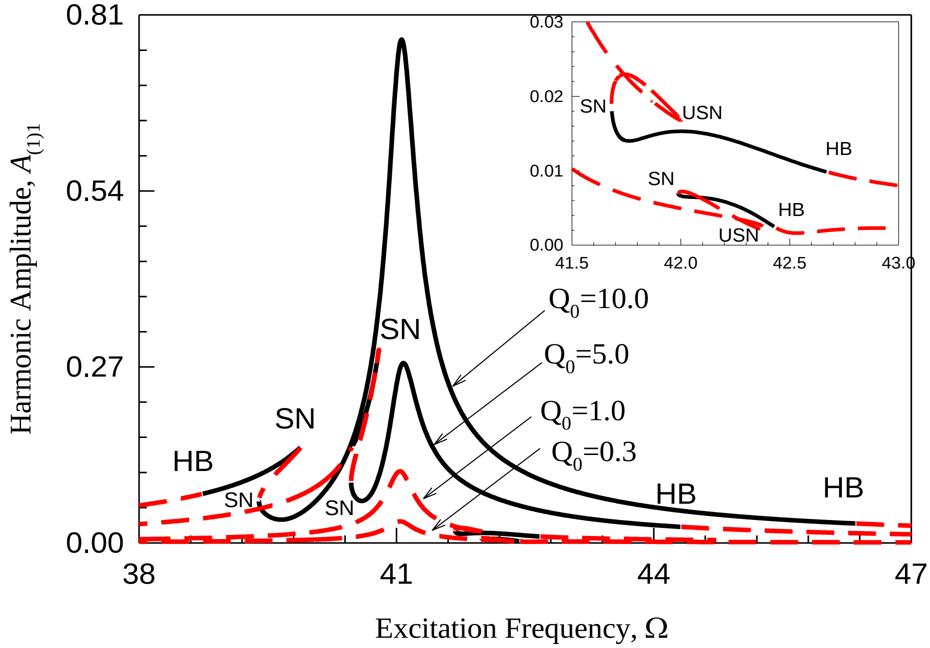
$$\begin{gathered} \zeta_y = 0.0037, \zeta_\theta = 0.0046, \eta_y = 0.000922, \eta_\theta = 0.0062 \\ a_1 = 2.341, a_3 = 14.366, b_1 = 0.496, b_3 = 1.265 \\ U = 6.77, \rho = 1.255, \end{gathered} \tag{4.61}$$

Where,

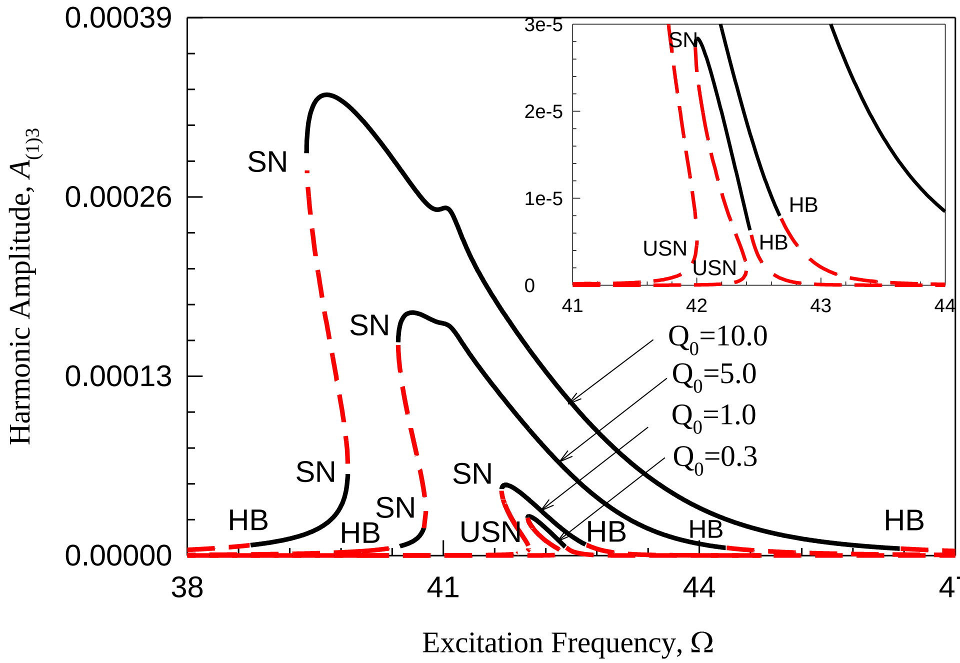
$$\begin{gathered} w_y = \sqrt{k_y / \mathfrak{M}},\ w_\theta = \sqrt{k_\theta / I},\ \zeta_y = c_y / 2\mathfrak{M} w_y, \\ \zeta_\theta = c_\theta / 2I w_\theta, \eta_y = \rho d^2 / 2\mathfrak{M}, \eta_\theta = \rho d^4 / 2I \end{gathered} \tag{4.62}$$

The acronym “SN” represents the saddle-node bifurcation. The acronym “HB” represents the Hopf bifurcation (supercritical). Solid and dashed curves represent stable and unstable period-1 motions, respectively. From the above the parameters, the frequency-amplitude curves of symmetric period-1 motion in both transverse and torsional directions are presented in Figures 4.1 and 4.2 that are based on 13 harmonic terms. The even number harmonic terms $A_{(i)k} = 0$ $(i = 1, 2\ k = 2, 4 \ldots 12)$. To consider periodic excitation amplitude effects on period-1 motion, the excitation amplitudes $Q_0 = 0.3, 1.0, 5.0, 10.0$ are selected. For $Q_0 = 0.3, 1.0$, there exists only one short range of stable period-1 motion, and the stable period-1 motions become quasi-periodic or chaos at the Hopf bifurcation (HB) $\Omega \approx 42.4288, 42.6688$ respectively. The jumping phenomena can be observed at the saddle node bifurcations (SN) $\Omega \approx 41.9876, 41.6823$ with excitation amplitude at $Q_0 = 0.3, 1.0$ respectively. As the amplitude of periodic excitation increases, the frequency range of the initial stable branch is expanding. Meanwhile another stable period-1 branch emerges in the relative low frequency area caused by switch of unstable saddle node bifurcation (USN) to stable saddle node bifurcation (SN). Another Hopf bifurcation appears on the left side of the newly existing stable period-1 branch at which the stable period-1 motion becomes quasi-period motions or chaos.

In Figures 4.1 and 4.2, scale types of the ordinates for harmonic amplitude $A_{(i)k}$ $(i=1,2$ $k=1,3)$ and $A_{(i)k}$ $(i=1,2\ k=5,7,9,13)$ are linear and logarithmic respectively. For the harmonic amplitudes $A_{(i)1}$ $(i=1,2)$, the bifurcation scenario can be seen clearly for excitation amplitudes Q_0=10.0, 5.0. In most frequency range, the magnitude of $A_{(i)1}$ $(i=1,2)$ for Q_0=10.0 are greater than Q_0=5.0. A zoomed window is provided for the lower excitation amplitude of Q_0 =1.0, 0.3. For $A_{(i)3}$ $(i=1,2)$, the quantity level is dropping with the decrease of excitation amplitude Q_0. A zoomed window is also provided for the stable range at Q_0=0.3. The quantity level of the first harmonic terms $A_{(i)1}$ $(i=1,2)$ for both directions are much great than the third harmonic terms $A_{(i)3}$ $(i=1,2)$ when Q_0 is the same. For harmonic amplitudes $A_{(i)k}$ $(i=1,2\quad k=5,7\ldots13)$, the quantity level is plunging as the excitation amplitude decreases. Bifurcation branches for different Q_0 can all be clearly observed. In addition, the pattern of how the harmonic amplitudes vary with changing Q_0 is very similar for different amplitudes $A_{(i)k}$ $(i=1,2\ k=5,7\ldots13)$. For last harmonic amplitudes $A_{(i)13}$ $(i=1,2)$, for different excitation amplitudes, they have been controlled under 10^{-12} and 10^{-11}. So the solutions for such a linear cable system are very accurate. Furthermore, the exact frequency value of the galloping vibration can be obtained if the excitation amplitude is kept decreasing.

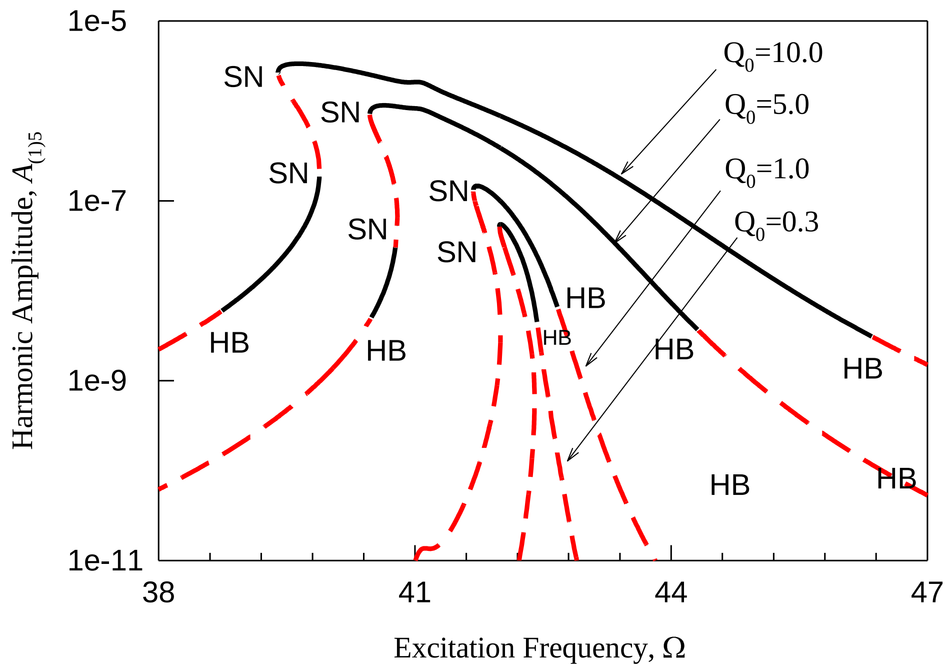


(i)



(ii)

Figure 4.1: Frequency-amplitude responses of period-1 transverse motions for linear cable based on 13 harmonic terms (HB13): (i) - (vii) $A_{(1)1} - A_{(1)13}$ ($\zeta_y = 0.0037, \zeta_\theta = 0.0046, \eta_y = 0.000922,$ $\eta_\theta = 0.0062, a_1 = 2.341, a_3 = 14.366, b_1 = 0.496, b_3 = 1.265, U = 6.77, \rho = 1.255, d = 33 \times 10^{-3},$ $Q_0 = 0.3, 1.0, 5.0, 10.0,$)



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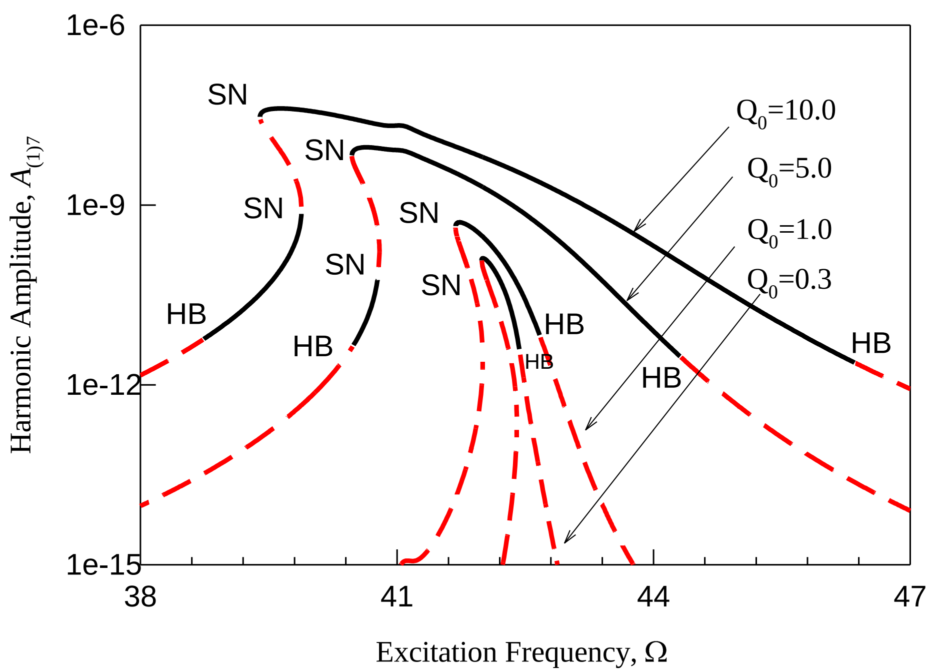
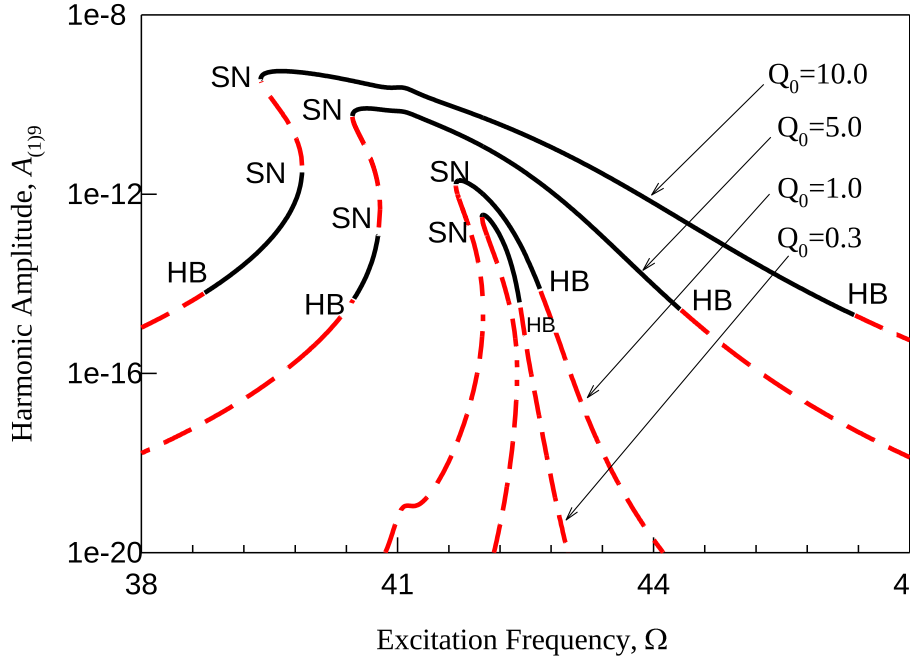


(iv)

Figure 4.1 Continued



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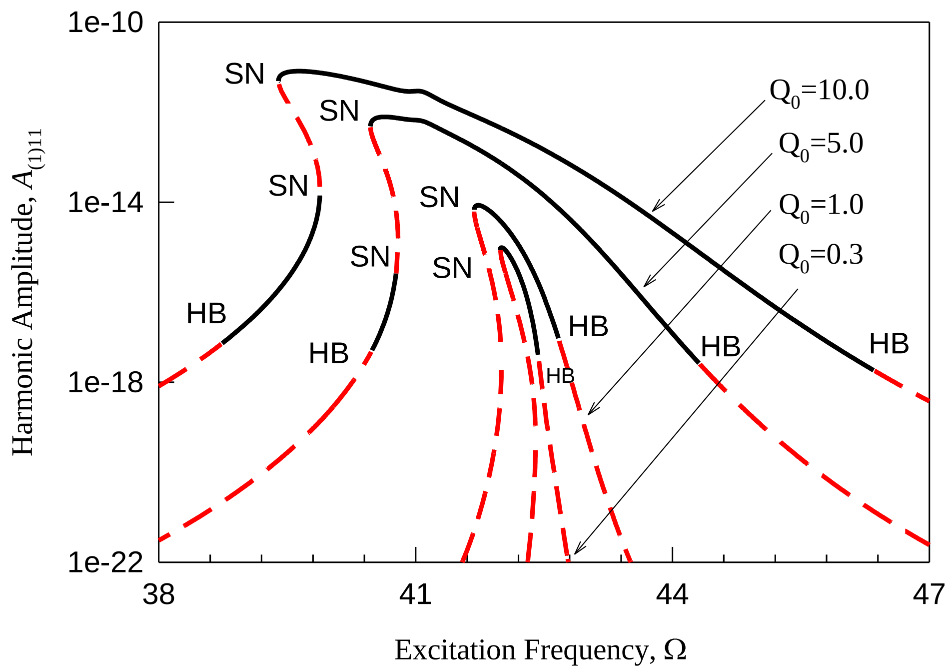

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Figure 4.1 Continued

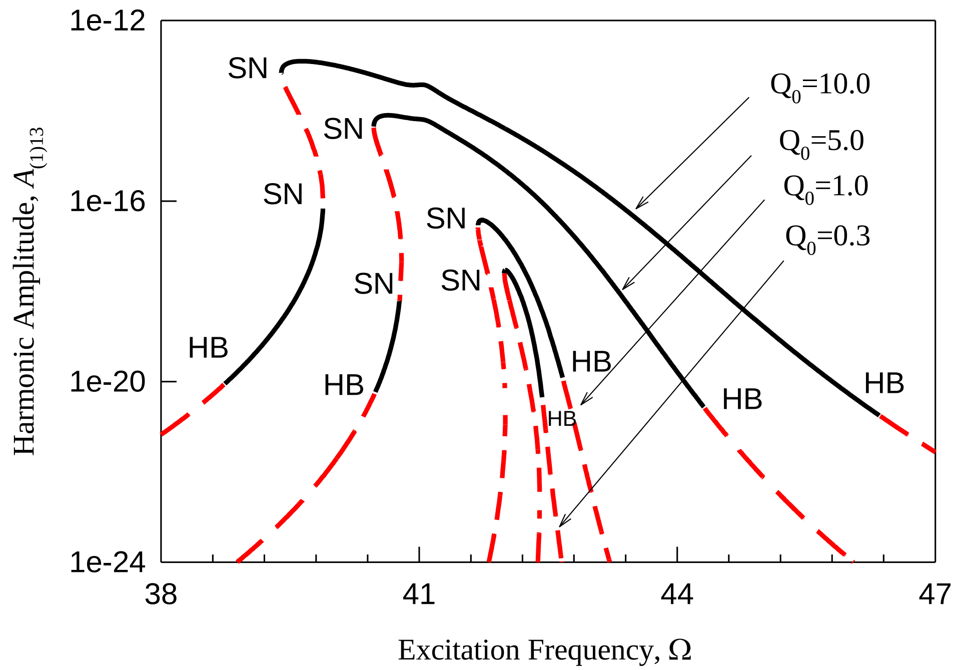

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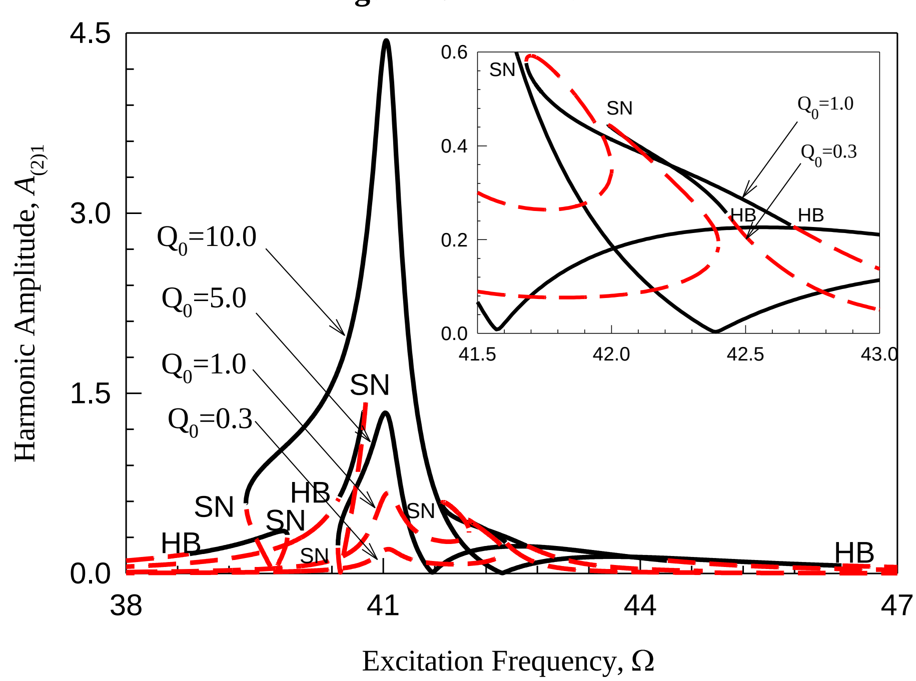
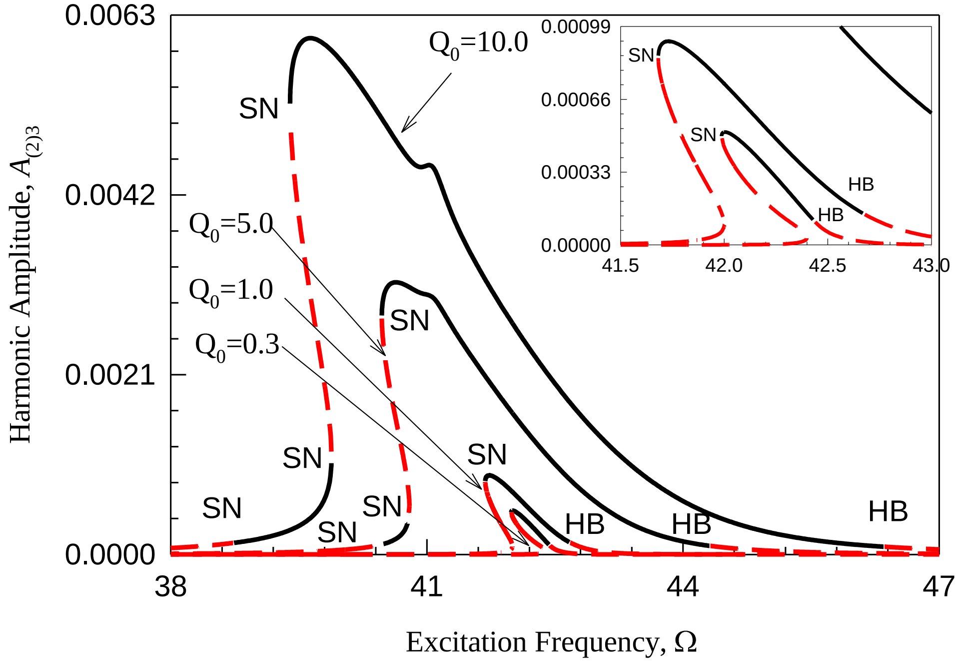
Figure 4.1 Continued



(i)

Figure 4.2: Frequency-amplitude responses of period-1 torsional motions for linear cable based on 13 harmonic terms (HB13): (i) - (vii) $A_{(2)1} - A_{(2)13}$ ($\zeta_y = 0.0037, \zeta_\theta = 0.0046, \eta_y = 0.000922,$ $\eta_\theta = 0.0062, a_1 = 2.341, a_3 = 14.366, b_1 = 0.496, b_3 = 1.265, U = 6.77, \rho = 1.255, d = 33\times10^{-3},$ $Q_0 = 0.3, 1.0, 5.0, 10.0,$)



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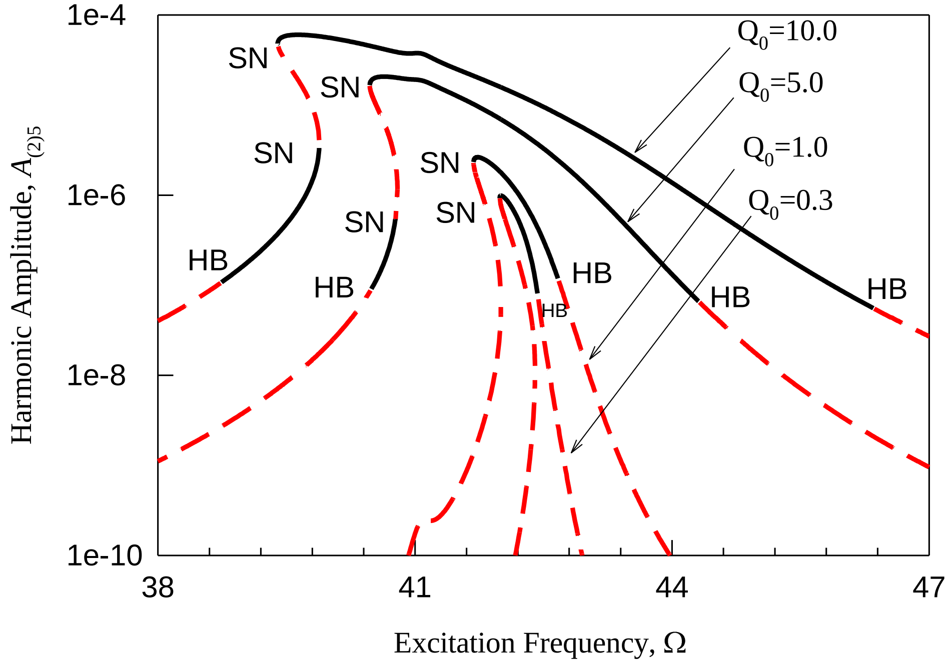

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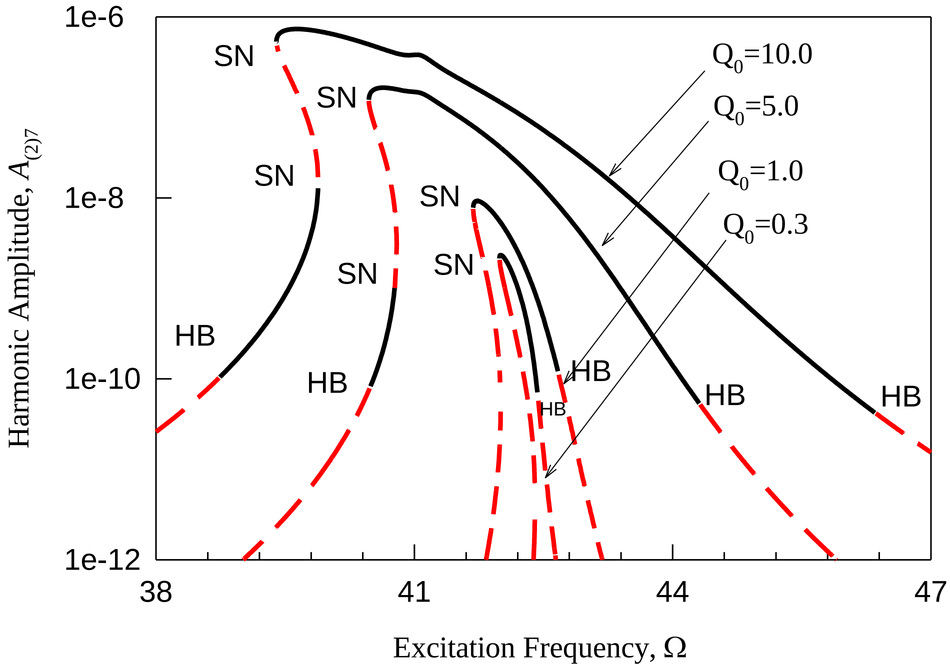
Figure 4.2 Continued



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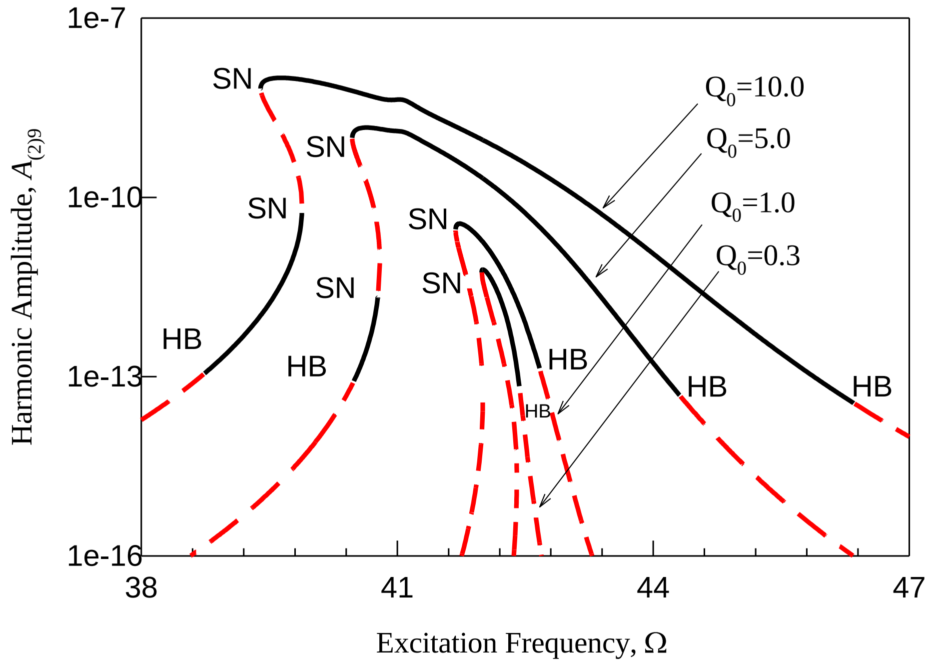

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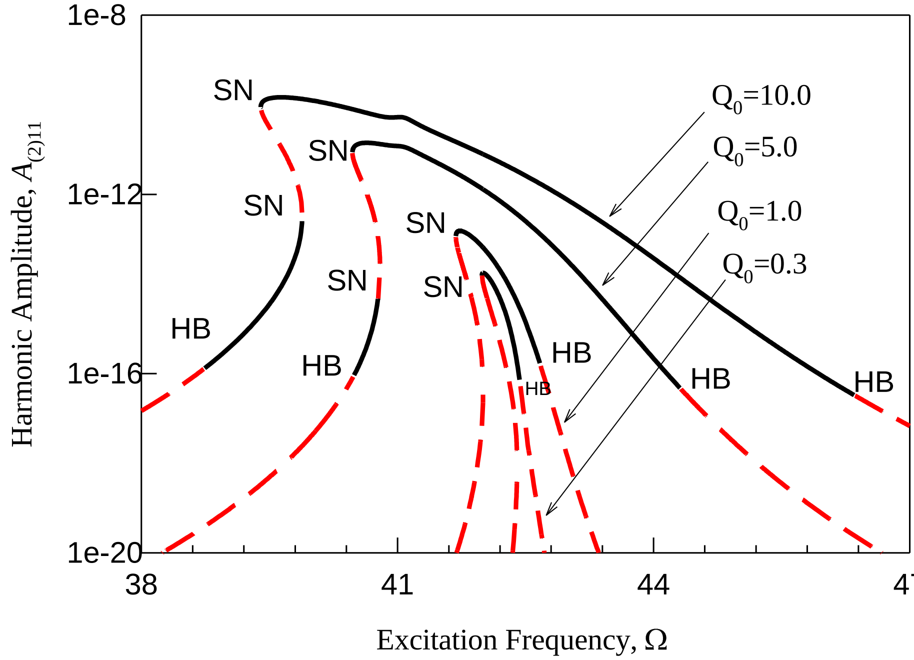
Figure 4.2 Continued



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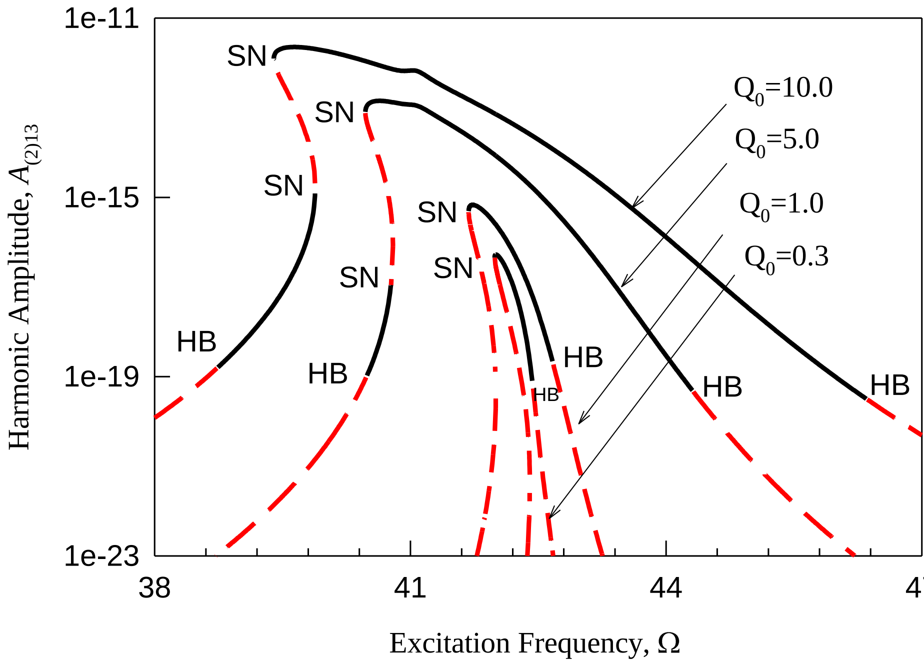
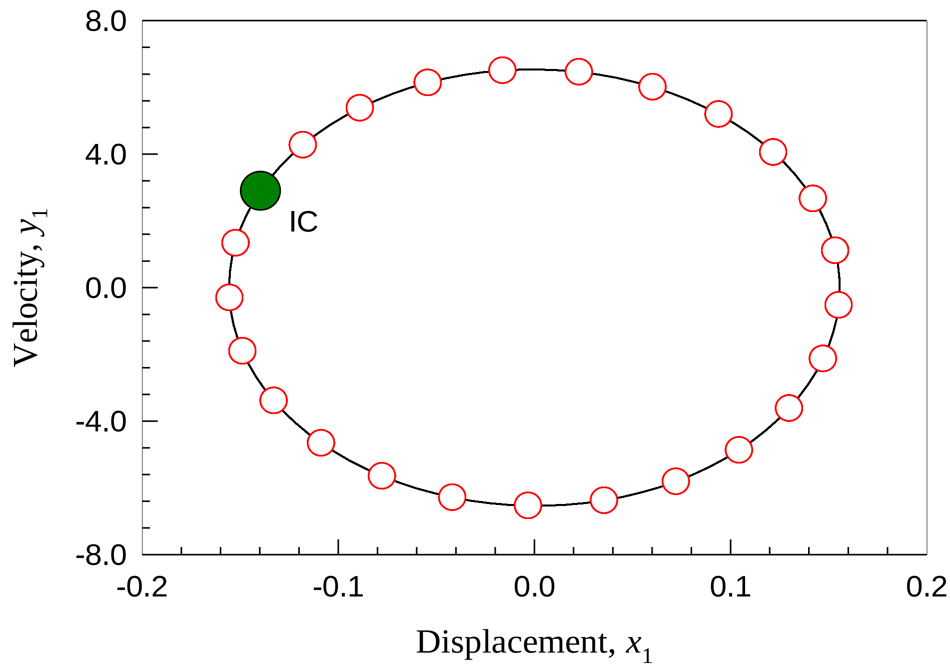


(vii)

Figure 4.2 Continued

4.3 Numerical Simulations

In comparison to periodic motion achieved by analytical solutions, numerical simulations for stable transverse and torsional motions under different excitation frequency are illustrated. The initial conditions for numerical simulations are computed from the approximate analytical solutions with 13 harmonic terms (HB13). Solid curves are used to represent the numerical results and the red circular symbols are analytical solutions. The initial condition is presented with green solid circle. The trajectory and harmonic amplitudes spectrum of stable symmetric period-1 motion will be presented in Figure 4.3 for $\Omega = 42.0, Q_0 = 10.0$ with initial conditions ($x_{10} \approx$ -0.13965 $\dot{x}_{10} \approx 2.89784$, $x_{20} \approx 0.11358$, $\dot{x}_{20} \approx 6.13986$). For over 200 periods, the analytical and numerical solutions match very well. The values of harmonic amplitudes for A_k , when k is even number, are all zeros. The twenty seventh harmonic term ($A_{(1)13}, A_{(2)13} \sim 10^{-15}$). In Figure 4.4, stable symmetrical period-1 motions in both directions at same frequency $\Omega = 12.0$ but different excitation amplitude $Q_0 = 5.0$ are presented. The 13 harmonic terms (HB13) are used. The initial conditions for both motions are ($x_{10} \approx$ -0.06990, $\dot{x}_{10} \approx 1.38305$) and ($x_{20} \approx$ -0.11035 $\dot{x}_{20} \approx$ -6.03975). The last harmonic term has $A_{(1)13}$, $A_{(2)13} \sim 10^{-15}$. The stable symmetric period-1 motions at about the same frequency $\Omega \approx 42.0$ under the smaller excitation amplitude of $Q_0 = 1.0, 0.3$ in Figure 4.5 are also computed with 13 harmonic terms (HB13) for both plunge and torsional motions. The initial conditions for motions in both directions are ($x_{10} \approx$ -0.01481, $\dot{x}_{10} \approx 0.11298$) and ($x_{20} \approx$ -0.23756, $\dot{x}_{20} \approx$ -12.82371) under $Q_0 = 1.0$. For the smaller $Q_0 = 0.3$, the initial conditions for both transverse and torsional motions are ($x_{10} \approx$ -6.47062e-3 $\dot{x}_{10} \approx 0.037446$) and ($x_{20} \approx$ -0.43224, $\dot{x}_{20} \approx 1.84272$). In order to better observe how the amplitude of periodic excitation affects the trajectories of both transverse and torsional motions, in Figure 4.6 the trajectories for different excitation amplitudes in each direction are all plotted together. The stable cycle in the plunge direction is shrinking as the amplitude of the periodic excitation decreases. In contrast, the orbit of the torsional motion is expanding with the decrease of excitation amplitude. Based on such phenomena, it can be postulated that the torsional motion dominates the transverse motion for the galloping vibration of such a two degree of freedom oscillator.



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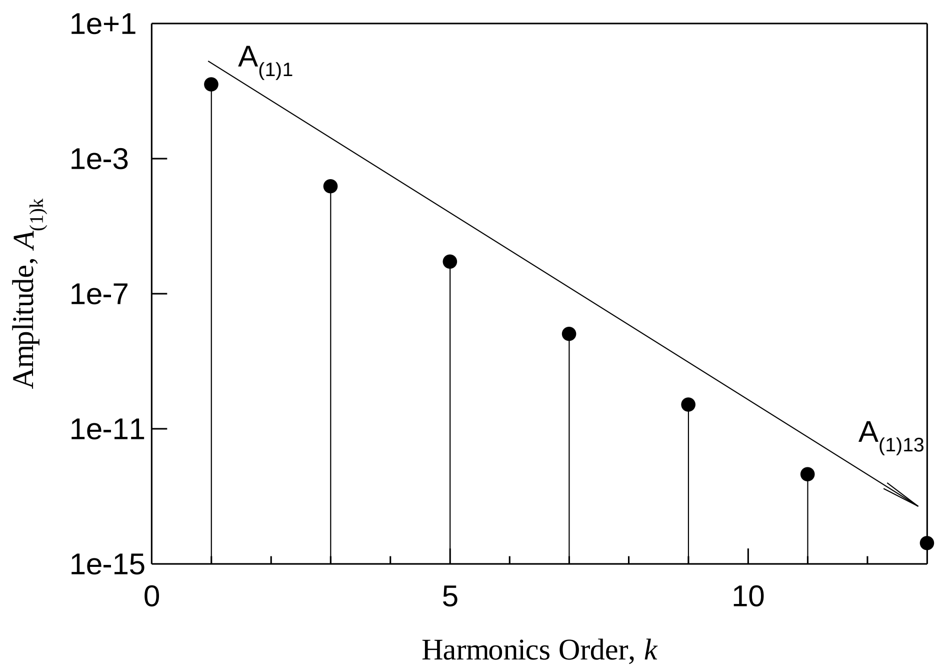
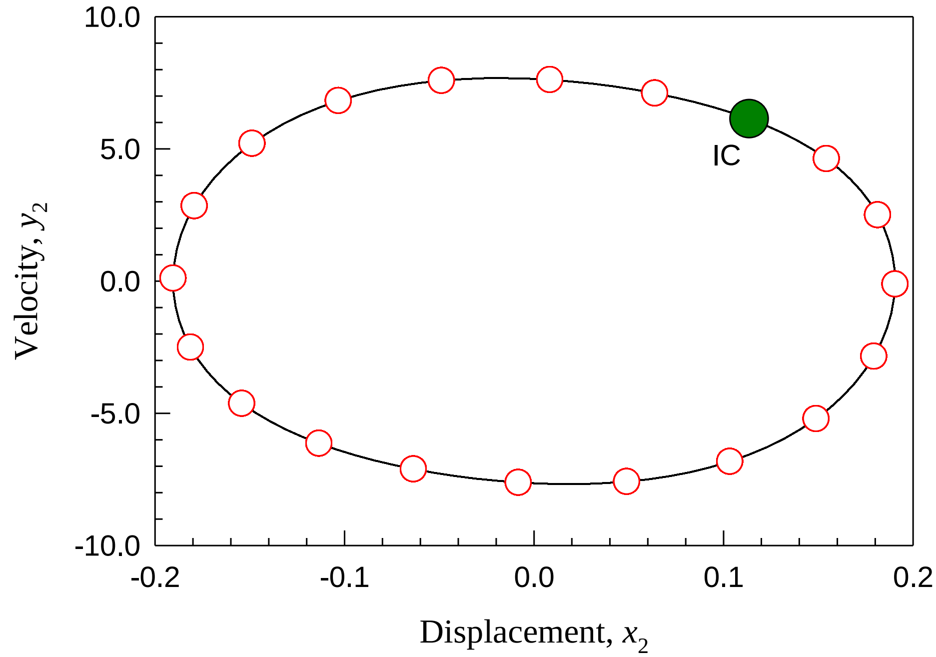


(ii)

Figure 4.3: Stable period-1 motion ($\Omega = 42.0$, HB13). The transverse motion with initial condition ($x_{10} \approx$ -0.13965 $\dot{x}_{10} \approx 2.89784,$): (i) trajectory and (ii) amplitude. The torsional motion with initial condition ($x_{20} \approx 0.11358$, $\dot{x}_{20} \approx 6.13986$): (iii) trajectory and (iv) amplitude. ($Q_0 = 10.0$).



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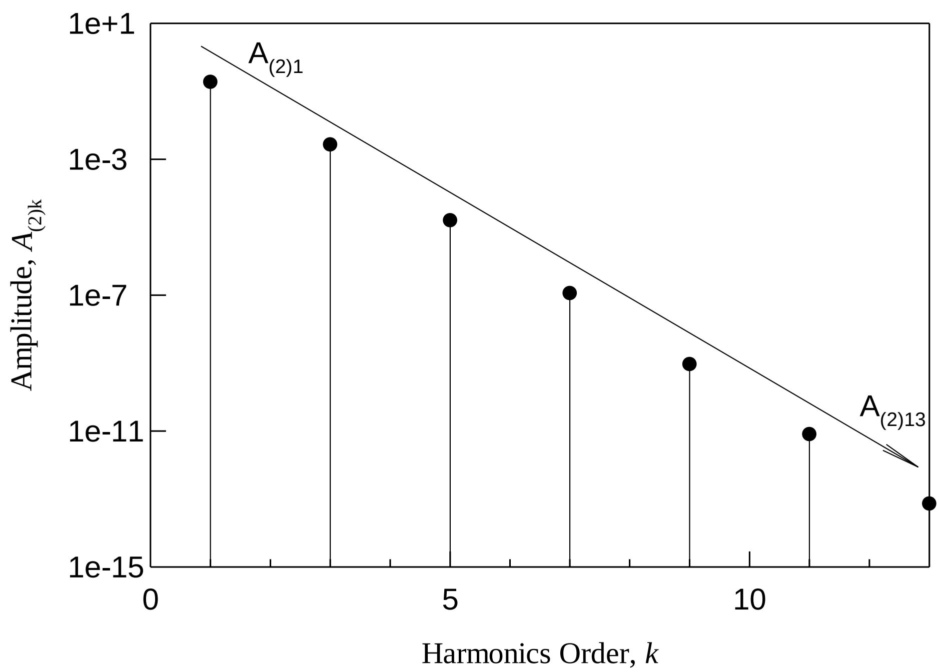

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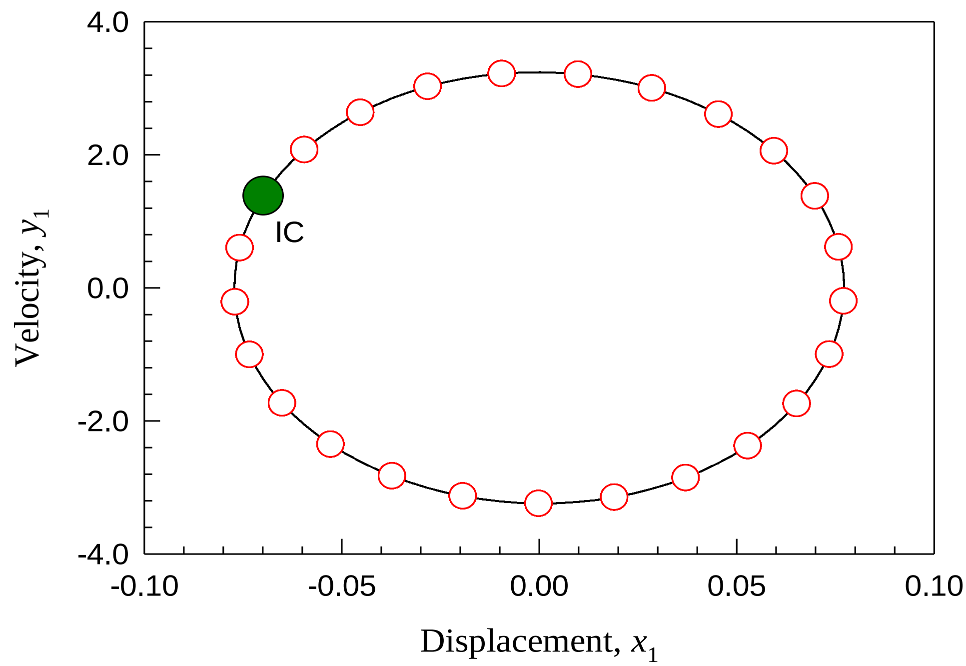

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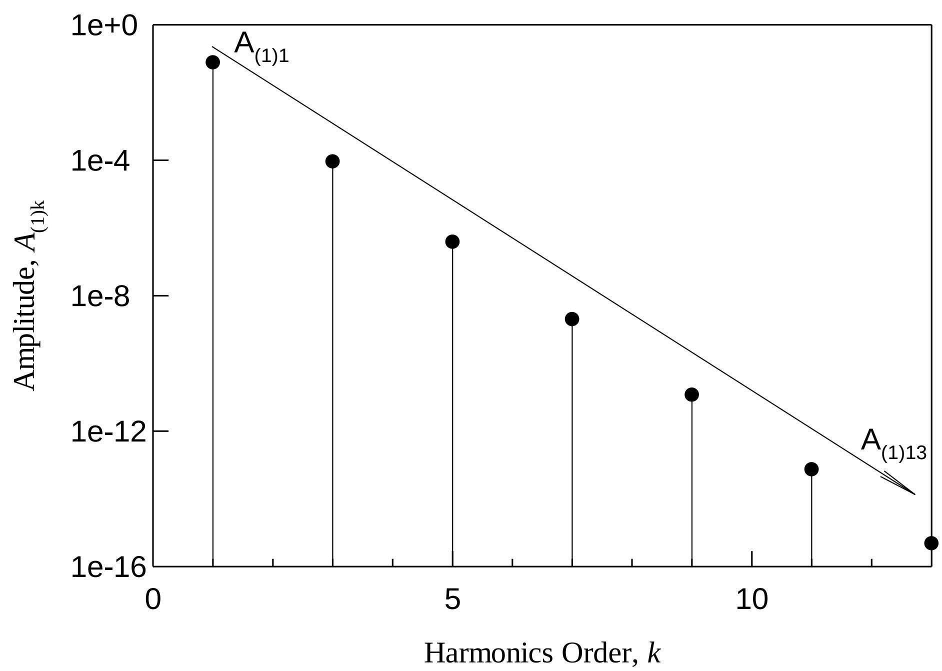
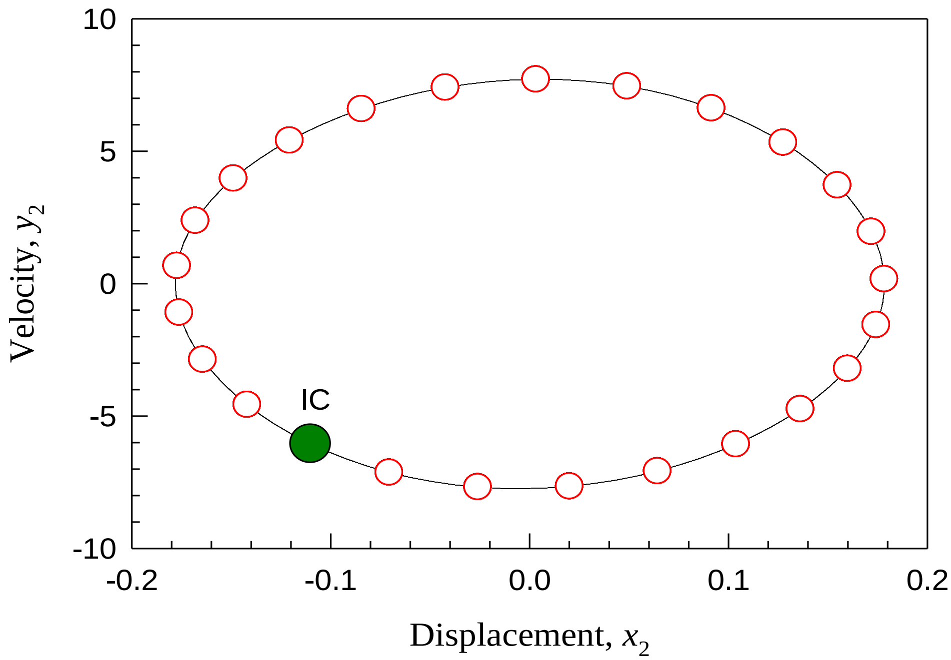


(ii)

Figure 4.4: Stable period-1 motion ($\Omega = 42.0$, HB13). The transverse motion with initial condition ($x_{10} \approx$ -0.06990, $\dot{x}_{10} \approx 1.38305$): (i) trajectory and (ii) amplitude. The torsional motion with initial condition ($x_{20} \approx$ -0.11035, $\dot{x}_{20} \approx$ -6.03975): (iii) trajectory and (iv) amplitude. ($Q_0 = 5.0$).



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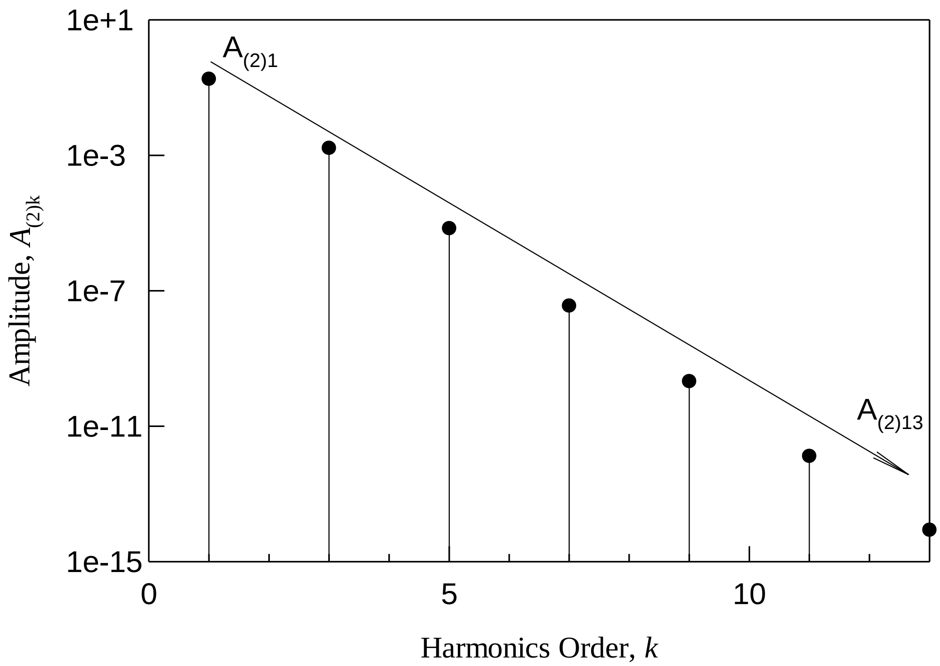

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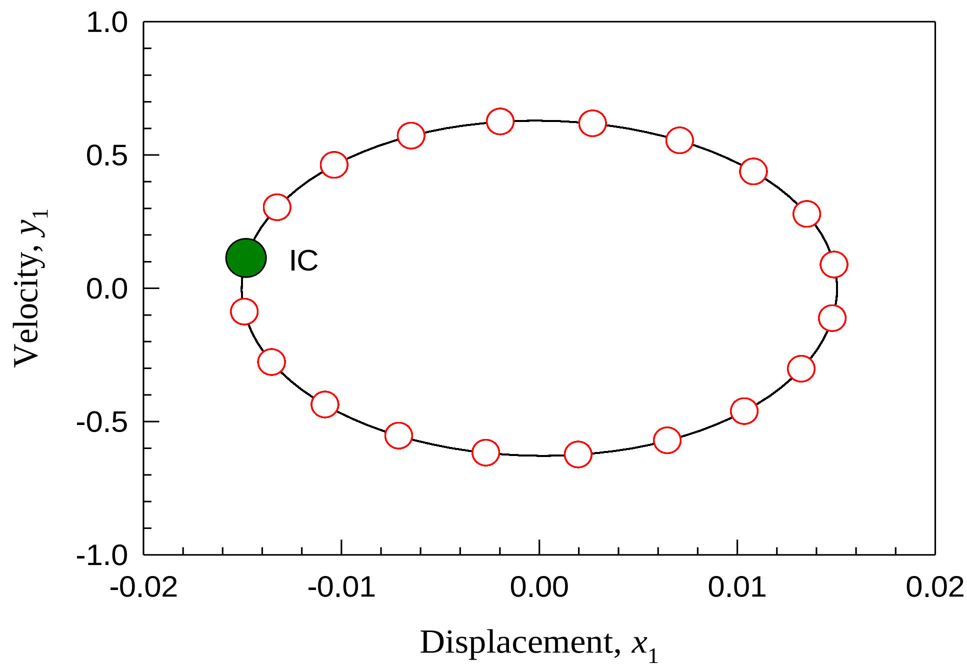
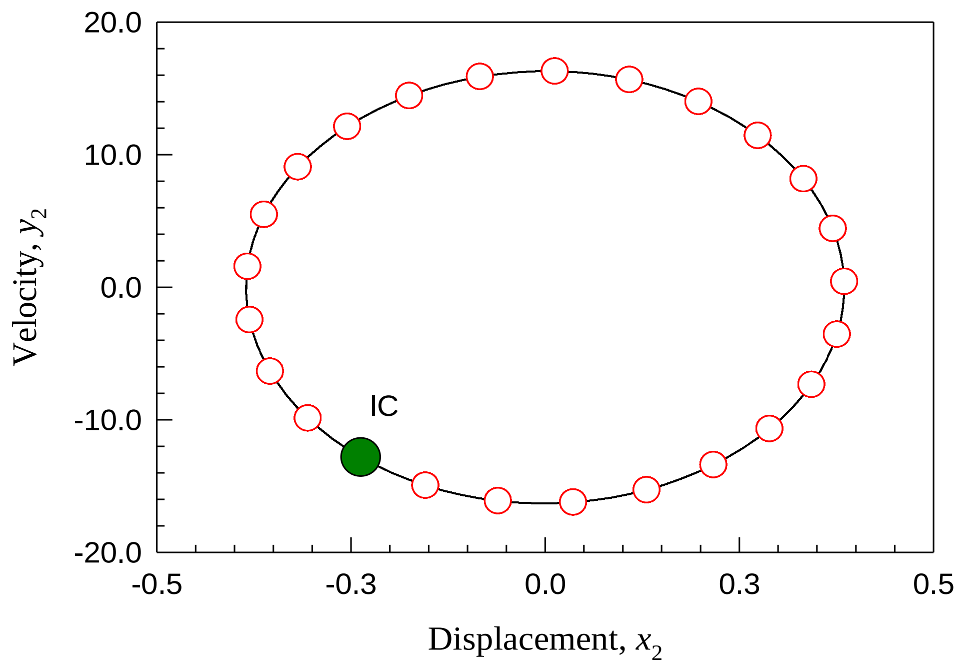
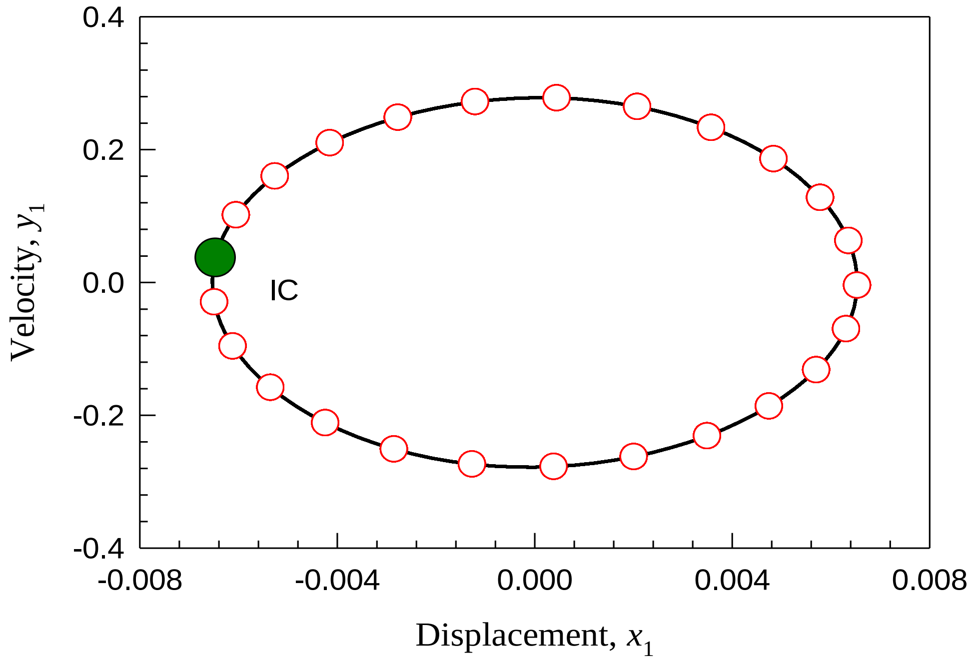


(i)



(ii)

Figure 4.5: Stable period-1 ($\Omega = 42.0$, HB13). The transverse and torsional motions motion with initial condition ($x_{10} \approx$ -0.01481, $\dot{x}_{10} \approx 0.11298$), ($x_{20} \approx$ -0.23756, $\dot{x}_{20} \approx$ -12.82371): (i) The transverse motion (ii) The torsional motion. ($Q_0 = 1.0$) The transverse motion and torsional motion with initial condition ($x_{10} \approx$ -6.47062e-3 $\dot{x}_{10} \approx 0.037446$), ($x_{20} \approx$ -0.43224, $\dot{x}_{20} \approx 1.84272$). (iii) The transverse motion and (iv) The torsional motion. ($Q_0 = 0.3$)



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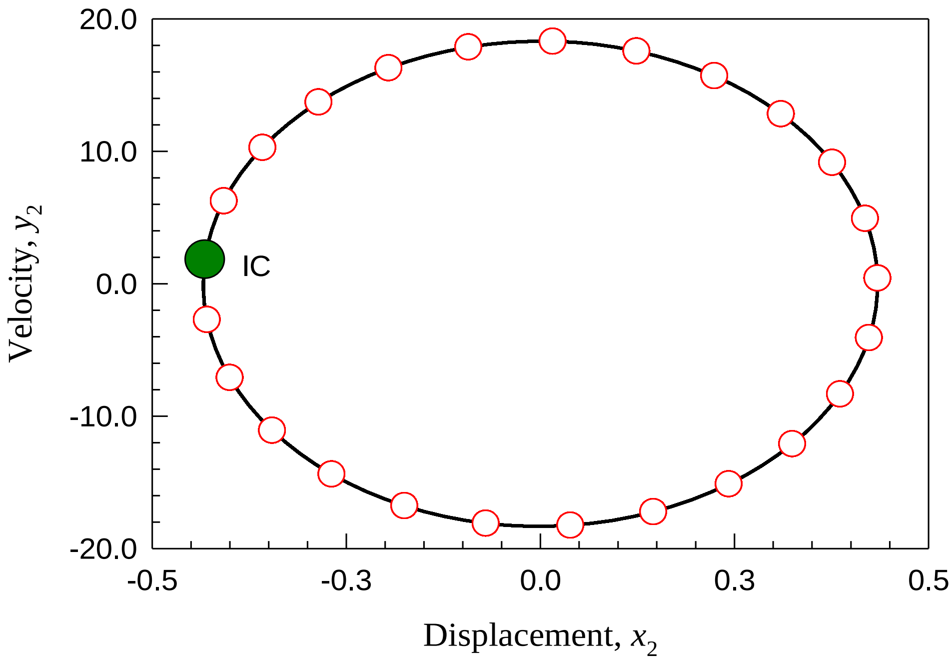

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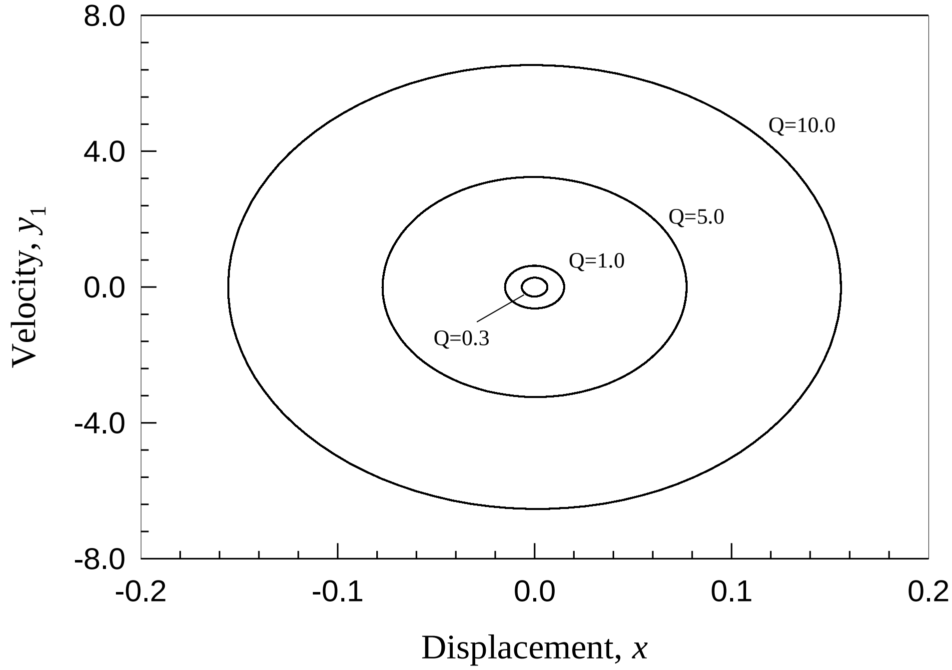
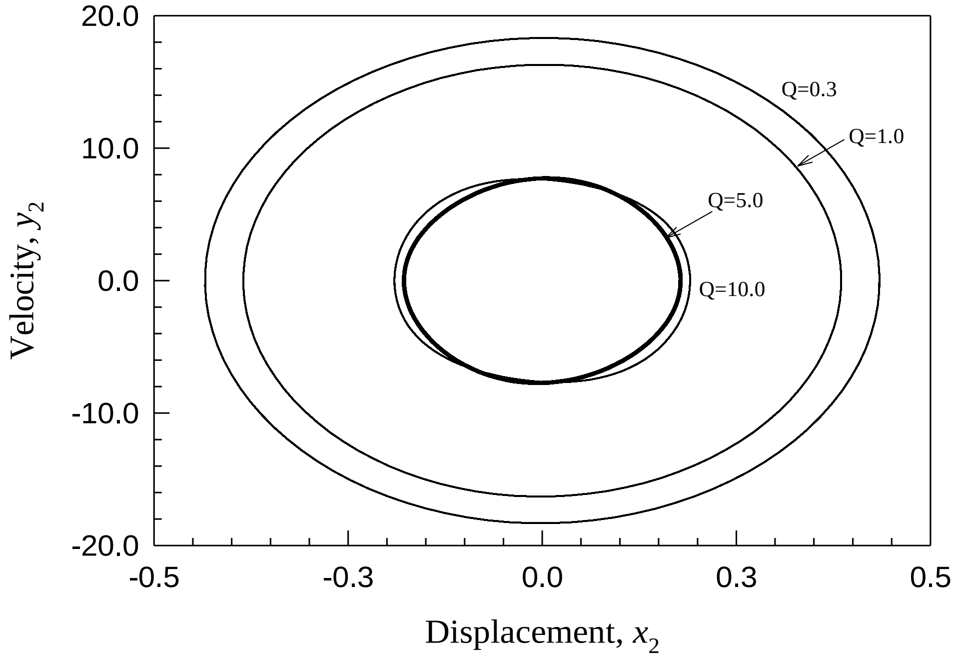
Figure 4.5 Continued



(i)



(ii)

Figure 4.6: Stable period-1 motion ($\Omega = 42.0$) under different excitation amplitude $Q_0 = 0.3$, 1.0, 5.0, 10.0, (i) The transverse motions and (ii) The torsional motions.

CHAPTER 5

ANALYTICAL SOLUTIONS FOR NONLINEAR CABLE

In this chapter, the periodic motion of a nonlinear cable structure with cubic nonlinear springs in both transverse and torsional directions are studied by using generalized harmonic balance method. The analytical bifurcation trees from period-m motions to chaos are obtained. The corresponding stability and bifurcation of the periodic motions are determined through the eigenvalue analysis. For a better understanding of complex period-m motions in such a two degree-of freedom cable system, numerical simulations of different periodic motions are illustrated. The harmonic amplitude spectra show the harmonic effects on periodic motions, and the corresponding accuracy of approximate analytical solutions can be observed.

5.1 Analytical Solutions

Consider a nonlinear cable structure with the same aerodynamics load and external load:

$$\begin{aligned} &\mathfrak{M}\ddot{v}(t)+c_y\dot{v}(t)+k_y v(t)+k_y^{'}v^3(t)=F_y+F_e \\ &I\ddot{\theta}(t)+c_\theta\dot{\theta}(t)+k_\theta\theta(t)+k_\theta^{'}\theta^3(t)=F_M \end{aligned} \tag{5.1}$$

Where $k_y^{'}$ coefficient of the transverse is nonlinear spring stiffness and $k_\theta^{'}$ is the coefficient of the torsional nonlinear spring stiffness. The aerodynamics load can be represented as

$$\begin{aligned} &F_y=\frac{1}{2}\rho U^2 d[-a_1(\theta-R_1\dot{\theta}/U-\dot{y}/U)+a_3(\theta-R_1\dot{\theta}/U-\dot{y}/U)^3] \\ &F_M=\frac{1}{2}\rho U^2 d^2[-b_1(\theta-R_1\dot{\theta}/U-\dot{y}/U)+b_3(\theta-R_1\dot{\theta}/U-\dot{y}/U)^3] \\ &F_e=Q_0\cos\Omega t \end{aligned} \tag{5.2}$$

The standard form of Eq. (5.1) can be written as

$$\ddot{\mathbf{x}} + \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, t) = \mathbf{0} \tag{5.3}$$

Where,

$$\mathbf{x} = (v, \theta)^{\mathrm{T}}, \dot{\mathbf{x}} = (\dot{v}, \dot{\theta})^{\mathrm{T}}, \mathbf{f} = (f_1, f_2)^{\mathrm{T}} \tag{5.4}$$

$$\begin{aligned} f_1 &= \alpha_1\dot{\theta}^3 + \alpha_2\dot{\theta}^2\dot{v} + \alpha_3\theta\dot{\theta}^2 + \alpha_4\dot{\theta}\dot{v}^2 + \alpha_5\theta\dot{\theta}\dot{v} + \alpha_6\theta^2\dot{\theta} + \alpha_7\dot{\theta} + \alpha_8\dot{v}^3 \\ &\quad + \alpha_9\theta\dot{v}^2 + \alpha_{10}\theta^2\dot{v} + \alpha_{11}\dot{v} + \alpha_{12}\theta^3 + \alpha_{13}\theta + \alpha_{14}\dot{v} + \alpha_{15}v + \alpha_{16}v^3 + Q\cos\Omega t \\ f_2 &= \beta_1\dot{\theta}^3 + \beta_2\dot{\theta}^2\dot{v} + \beta_3\theta\dot{\theta}^2 + \beta_4\dot{\theta}\dot{v}^2 + \beta_5\theta\dot{\theta}\dot{v} + \beta_6\theta^2\dot{\theta} + \beta_7\dot{\theta} + \beta_8\dot{v}^3 \\ &\quad + \beta_9\theta\dot{v}^2 + \beta_{10}\theta^2\dot{v} + \beta_{11}\dot{v} + \beta_{12}\theta^3 + \beta_{13}\theta + \beta_{14}\dot{\theta} + \beta_{15}\theta + \beta_{16}\theta^3 \end{aligned} \tag{5.5}$$

And

$$\begin{aligned} &\alpha_1 = \frac{\rho dR^3 a_3}{2U\mathfrak{M}}\ \alpha_2 = \frac{3\rho dR^2 a_3}{2U\mathfrak{M}}, \alpha_3 = -\frac{3\rho dR^2 a_3}{2\mathfrak{M}}, \alpha_4 = \frac{3\rho dRa_3}{2U\mathfrak{M}}, \alpha_5 = -\frac{6\rho dRa_3}{2\mathfrak{M}}, \\ &\alpha_6 = \frac{3\rho dRUa_3}{2\mathfrak{M}}, \alpha_7 = -\frac{\rho dRUa_1}{2\mathfrak{M}}, \alpha_8 = \frac{\rho da_3}{2U\mathfrak{M}}, \alpha_9 = -\frac{3\rho da_3}{2\mathfrak{M}}, \alpha_{10} = \frac{3\rho dUa_3}{2\mathfrak{M}}, \\ &\alpha_{11} = -\frac{\rho dUa_1}{2\mathfrak{M}}, \alpha_{12} = -\frac{\rho dU^2 a_3}{2\mathfrak{M}}, \alpha_{13} = \frac{\rho dU^2 a_1}{2\mathfrak{M}}, \alpha_{14} = \frac{c_y}{\mathfrak{M}}, \alpha_{15} = \frac{k_y}{\mathfrak{M}}, \\ &\alpha_{16} = \frac{k_y^{'}}{\mathfrak{M}}, \\ &Q = -\frac{Q_0}{\mathfrak{M}}, \beta_1 = \frac{\rho d^2 R^3 b_3}{2UI}\ \beta_2 = \frac{3\rho d^2 R^2 b_3}{2UI}, \beta_3 = -\frac{3\rho d^2 R^2 b_3}{2I}, \beta_4 = \frac{3\rho d^2 Rb_3}{2UI}, \\ &\beta_5 = -\frac{6\rho d^2 Rb_3}{2I}, \beta_6 = \frac{3\rho d^2 RUb_3}{2I}, \beta_7 = -\frac{\rho d^2 RUb_1}{2I}, \beta_8 = \frac{\rho d^2 b_3}{2UI}, \\ &\beta_9 = -\frac{3\rho d^2 b_3}{2I}, \beta_{10} = \frac{3\rho d^2 Ub_3}{2I}, \beta_{11} = -\frac{\rho d^2 Ub_1}{2I}, \beta_{12} = -\frac{\rho d^2 U^2 b_3}{2I}, \\ &\beta_{13} = \frac{\rho d^2 U^2 b_1}{2I}, \beta_{14} = \frac{c_\theta}{I}, \beta_{15} = \frac{k_\theta}{I}, \beta_{16} = \frac{k_\theta^{'}}{I} \end{aligned} \tag{5.6}$$

The analytical solution of period-m motion for the above equations are

$$\begin{aligned} v^*(t) &= a_{10}^{(m)}(t) + \sum_{k=1}^{N} b_{1k/m}(t)\cos(\frac{k\Omega t}{m}) + c_{1k/m}(t)\sin(\frac{k\Omega t}{m}), \\ \theta^*(t) &= a_{20}^{(m)}(t) + \sum_{k=1}^{N} b_{2k/m}(t)\cos(\frac{k\Omega t}{m}) + c_{2k/m}(t)\sin(\frac{k\Omega t}{m}); \end{aligned} \tag{5.7}$$

Then the first and second order derivatives of $v^*(t)$ and $\theta^*(t)$ are

$$\dot{v}^*(t) = \dot{a}_{10}^{(m)} + \sum_{k=1}^{N} (\dot{b}_{1k/m} + \frac{k\Omega}{m} c_{1k/m}) \cos(\frac{k\Omega}{m} t) + (\dot{c}_{1k/m} - \frac{k\Omega}{m} b_{1k/m}) \sin(\frac{k\Omega}{m} t),$$
$$= \dot{a}_{10}^{(m)} + \sum_{k=1}^{N} P_{k/m} \cos(\frac{k\Omega}{m} t) + Q_{k/m} \sin(\frac{k\Omega}{m} t),$$
$$\dot{\theta}^*(t) = \dot{a}_{20}^{(m)} + \sum_{k=1}^{N} (\dot{b}_{2k/m} + \frac{k\Omega}{m} c_{2k/m}) \cos(\frac{k\Omega}{m} t) + (\dot{c}_{2k/m} - \frac{k\Omega}{m} b_{2k/m}) \sin(\frac{k\Omega}{m} t);$$
$$= \dot{a}_{20}^{(m)} + \sum_{k=1}^{N} B_{k/m} \cos(\frac{k\Omega}{m} t) + C_{k/m} \sin(\frac{k\Omega}{m} t); \tag{5.8}$$
$$\ddot{v}^*(t) = \ddot{a}_{10}^{(m)} + \sum_{k=1}^{N} [\ddot{b}_{1k/m} + 2\frac{k\Omega}{m} \dot{c}_{1k/m} - (\frac{k\Omega}{m})^2 b_{1k/m}] \cos(\frac{k\Omega}{m} t)$$
$$+ [\ddot{c}_{1k/m} - 2\frac{k\Omega}{m} \dot{b}_{1k/m} - (\frac{k\Omega}{m})^2 c_{1k/m}] \sin(\frac{k\Omega}{m} t),$$
$$\ddot{\theta}^*(t) = \ddot{a}_{20}^{(m)} + \sum_{k=1}^{N} [\ddot{b}_{2k/m} + 2\frac{k\Omega}{m} \dot{c}_{2k/m} - (\frac{k\Omega}{m})^2 b_{2k/m}] \cos(\frac{k\Omega}{m} t)$$
$$+ [\ddot{c}_{2k/m} - 2\frac{k\Omega}{m} \dot{b}_{2k/m} - (\frac{k\Omega}{m})^2 c_{2k/m}] \sin(\frac{k\Omega}{m} t).$$

where

$$B_{i/m} = \dot{b}_{2i/m} + i\Omega c_{2i/m} / m,\ C_{i/m} = \dot{c}_{2i/m} - i\Omega b_{2i/m} / m$$
$$P_{i/m} = \dot{b}_{1i/m} + i\Omega c_{1i/m} / m,\ Q_{i/m} = \dot{c}_{1i/m} - i\Omega b_{1i/m} / m \tag{5.9}$$

Define

$$\mathbf{a}_0^{(m)} = (a_{10}^{(m)}, a_{20}^{(m)})^{\mathrm{T}},$$
$$\mathbf{b}^{(m)} = (b_{11/m}, b_{12/m}, \cdots, b_{1N/m}, b_{21/m}, b_{22/m}, \cdots, b_{2N/m})^{\mathrm{T}}$$
$$= (\mathbf{b}_1^{(m)}; \mathbf{b}_2^{(m)}). \tag{5.10}$$
$$\mathbf{c}^{(m)} = (c_{11/m}, c_{12/m}, \cdots, c_{1N/m}, c_{21/m}, c_{22/m}, \cdots, c_{2N/m})^{\mathrm{T}}$$
$$= (\mathbf{c}_1^{(m)}; \mathbf{c}_2^{(m)}).$$

Substitution of Eqs.(5.8), (5.9) into Eq.(5.3) and averaging for the harmonic terms of $\cos(k\Omega t / m)$ and $\sin(k\Omega t / m)$ $(k = 0,1,2,\ldots)$ gives

$$
\begin{aligned}
&\ddot{a}_{10}^{(m)} + F_{10}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0\\
&\ddot{b}_{1k/m} + 2\frac{k\Omega}{m}\dot{c}_{1k/m} - (\frac{k\Omega}{m})^2 b_{1k/m} + F_{1k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0,\\
&\ddot{c}_{1k/m} - 2\frac{k\Omega}{m}\dot{b}_{1k/m} - (\frac{k\Omega}{m})^2 c_{1k/m} + F_{1k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0;\\
&\ddot{a}_{20}^{(m)} + F_{20}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0,\\
&\ddot{b}_{2k/m} + 2k\Omega\dot{c}_{2k/m} - (\frac{k\Omega}{m})^2 b_{2k/m} + F_{2k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0,\\
&\ddot{c}_{2k/m} - 2k\Omega\dot{b}_{2k/m} - (\frac{k\Omega}{m})^2 c_{2k/m} + F_{2k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) = 0.
\end{aligned}
\tag{5.11}
$$

Where

$$
\begin{aligned}
F_{10}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{1}{mT}\int_0^{mT} f_1(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)dt\\
F_{1k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{2}{mT}\int_0^{mT} f_1(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)\cos(\frac{k\Omega}{m}t)dt\\
F_{1k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{2}{mT}\int_0^{mT} f_1(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)\sin(\frac{k\Omega}{m}t)dt\\
F_{20}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{1}{mT}\int_0^{mT} f_2(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)dt\\
F_{2k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{2}{mT}\int_0^{mT} f_2(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)\cos(\frac{k\Omega}{m}t)dt\\
F_{2k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \frac{2}{mT}\int_0^{mT} f_2(\mathbf{x}^{(m)*},\dot{\mathbf{x}}^{(m)*},t)\sin(\frac{k\Omega}{m}t)dt
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
F_{10}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\alpha_i f_1^i + \alpha_{14} f_{11}^{14} + \alpha_{15} f_{11}^{15} + \alpha_{16} f_{11}^{16}\\
F_{1k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\alpha_i f_2^i + \alpha_{14} f_{21}^{14} + \alpha_{15} f_{21}^{15} + Qf_1^{16} + \alpha_{16} f_{21}^{16}\\
F_{1k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\alpha_i f_3^i + \alpha_{14} f_{31}^{14} + \alpha_{15} f_{31}^{15} + \alpha_{16} f_{31}^{16}\\
F_{20}^{(m)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\beta_i f_1^i + \beta_{14} f_{12}^{14} + \beta_{15} f_{12}^{15} + \beta_{16} f_{12}^{16}\\
F_{2k/m}^{(c)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\beta_i f_2^i + \beta_{14} f_{22}^{14} + \beta_{15} f_{22}^{15} + \beta_{16} f_{22}^{16}\\
F_{2k/m}^{(s)}(\mathbf{a}_0^{(m)},\mathbf{b}^{(m)},\mathbf{c}^{(m)},\dot{\mathbf{a}}_0^{(m)},\dot{\mathbf{b}}_0^{(m)},\dot{\mathbf{c}}_0^{(m)}) &= \sum_{i=1}^{13}\beta_i f_3^i + \beta_{14} f_{32}^{14} + \beta_{15} f_{32}^{15} + \beta_{16} f_{32}^{16}
\end{aligned}
\tag{5.13}
$$

Define

$$\begin{aligned}
\mathbf{z}^{(m)} &= (a_{10}^{(m)}, b_{11/m}, \cdots, b_{1N/m}, c_{11/m}, \cdots, c_{1N/m}; a_{20}^{(m)}, b_{21/m}, \cdots, b_{2N/m}, c_{21/m}, \cdots, c_{2N/m})^{\mathrm{T}} \\
&\equiv (z_1^{(m)}, z_2^{(m)}, \cdots, z_{2N+1}^{(m)}; z_{2N+2}^{(m)}, z_{2N+3}^{(m)}, \cdots, z_{4N+2}^{(m)})^{\mathrm{T}} \\
\mathbf{z}_1^{(m)} \triangleq \dot{\mathbf{z}}^{(m)} &= (\dot{a}_{10}^{(m)}, \dot{b}_{11/m}, \cdots, \dot{b}_{1N/m}, \dot{c}_{11/m}, \cdots, \dot{c}_{1N/m}; \dot{a}_{20}^{(m)}, \dot{b}_{21/m}, \cdots, \dot{b}_{2N/m}, \dot{c}_{21/m}, \cdots, \dot{c}_{2N/m})^{\mathrm{T}} \\
&\equiv (\dot{z}_1^{(m)}, \dot{z}_2^{(m)}, \cdots, \dot{z}_{2N+1}^{(m)}; \dot{z}_{2N+2}^{(m)}, \dot{z}_{2N+3}^{(m)}, \cdots, \dot{z}_{4N+2}^{(m)})^{\mathrm{T}}
\end{aligned} \tag{5.14}$$

Equations (5.11) can be rewritten as

$$\dot{\mathbf{z}}^{(m)} = \mathbf{z}_1^{(m)} \text{ and } \dot{\mathbf{z}}_1^{(m)} = \mathbf{g}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \tag{5.15}$$

where

$$\mathbf{g}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) = \begin{pmatrix}
-F_{10}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \\
-\mathbf{F}_{1/m}^{(c)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) - 2\dfrac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{c}}_1^{(m)} + \mathbf{k}_2(\dfrac{\Omega}{m})^2\mathbf{b}_1^{(m)} \\
-\mathbf{F}_{1/m}^{(s)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) + 2\dfrac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{b}}_1^{(m)} + \mathbf{k}_2(\dfrac{\Omega}{m})^2\mathbf{c}_1^{(m)} \\
-F_{20}^{(m)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \\
-\mathbf{F}_{2/m}^{(c)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) - 2\dfrac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{c}}_2^{(m)} + \mathbf{k}_2(\dfrac{\Omega}{m})^2\mathbf{b}_2^{(m)} \\
-\mathbf{F}_{2/m}^{(s)}(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) + 2\dfrac{\mathbf{k}_1\Omega}{m}\dot{\mathbf{b}}_2^{(m)} + \mathbf{k}_2(\dfrac{\Omega}{m})^2\mathbf{c}_2^{(m)}
\end{pmatrix} \tag{5.16}$$

where

$$\begin{aligned}
&\mathbf{k}_1 = diag(1, 2, \cdots, N), \\
&\mathbf{k}_2 = diag(1, 2^2, \cdots, N^2), \\
&\mathbf{F}_{1/m}^{(c)} = (F_{11/m}^{(c)}, F_{12/m}^{(c)}, \cdots, F_{1N/m}^{(s)})^{\mathrm{T}}, \\
&\mathbf{F}_{1/m}^{(s)} = (F_{11/m}^{(s)}, F_{12/m}^{(s)}, \cdots, F_{1N/m}^{(s)})^{\mathrm{T}}, \\
&\mathbf{F}_{2/m}^{(c)} = (F_{21/m}^{(c)}, F_{22/m}^{(c)}, \cdots, F_{2N/m}^{(c)})^{\mathrm{T}}, \\
&\mathbf{F}_{2/m}^{(s)} = (F_{21/m}^{(s)}, F_{22/m}^{(s)}, \cdots, F_{2N/m}^{(s)})^{\mathrm{T}} \\
&\text{for } N = 1, 2, \cdots, \infty.
\end{aligned} \tag{5.17}$$

Setting

$$\mathbf{y}^{(m)} \equiv (\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}) \text{ and } \mathbf{f}^{(m)} = (\mathbf{z}_1^{(m)}, \mathbf{g}^{(m)})^{\mathrm{T}}, \tag{5.18}$$

Thus, equation (5.12) becomes

$$\dot{\mathbf{y}}^{(m)} = \mathbf{f}^{(m)}(\mathbf{y}^{(m)}). \tag{5.19}$$

The steady-state solutions for periodic motion can be obtained by setting $\dot{\mathbf{y}}^{(m)} = \mathbf{0}$, i.e.,

$$\begin{aligned}
&F_{10}^{(m)}(\mathbf{z}^{(m)}, \mathbf{0}) = 0 \\
&-\mathbf{F}_{1/m}^{(c)}(\mathbf{z}^{(m)}, \mathbf{0}) + \mathbf{k}_2(\frac{\Omega}{m})^2 \mathbf{b}_1^{(m)} = \mathbf{0} \\
&-\mathbf{F}_{1/m}^{(s)}(\mathbf{z}^{(m)}, \mathbf{0}) + \mathbf{k}_2(\frac{\Omega}{m})^2 \mathbf{c}_1^{(m)} = \mathbf{0} \\
&F_{20}^{(m)}(\mathbf{z}^{(m)}, \mathbf{0}) = 0 \\
&-\mathbf{F}_{2/m}^{(c)}(\mathbf{z}^{(m)}, \mathbf{0}) + \mathbf{k}_2(\frac{\Omega}{m})^2 \mathbf{b}_2^{(m)} = \mathbf{0} \\
&-\mathbf{F}_{2/m}^{(s)}(\mathbf{z}^{(m)}, \mathbf{0}) + \mathbf{k}_2(\frac{\Omega}{m})^2 \mathbf{c}_2^{(m)} = \mathbf{0}
\end{aligned} \tag{5.20}$$

The $(4N+2)$ nonlinear equations in Eq.(5.20) are solved by the Newton-Raphson method. In Luo [2012], the linearized equation at $\mathbf{y}^{(m)*} = (\mathbf{z}^{(m)*}, \mathbf{0})^{\mathrm{T}}$ is

$$\Delta\dot{\mathbf{y}}^{(m)} = D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)})\Delta\mathbf{y}^{(m)} \tag{5.21}$$

where

$$D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)}) = \partial\mathbf{f}^{(m)}(\mathbf{y}^{(m)})/\partial\mathbf{y}^{(m)}\Big|_{\mathbf{y}^{(m)*}} \tag{5.22}$$

The corresponding eigenvalues are determined by

$$\left| D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)}) - \lambda\mathbf{I}_{4(2N+1)\times 4(2N+1)} \right| = 0. \tag{5.23}$$

where

$$D\mathbf{f}(\mathbf{y}^{(m)*}) = \begin{bmatrix} \mathbf{0}_{2(2N+1)\times 2(2N+1)} & \mathbf{I}_{2(2N+1)\times 2(2N+1)} \\ \mathbf{G}_{2(2N+1)\times 2(2N+1)} & \mathbf{H}_{2(2N+1)\times 2(2N+1)} \end{bmatrix} \tag{5.24}$$

and

$$\mathbf{G}=\frac{\partial \mathbf{g}^{(m)}}{\partial \mathbf{z}^{(m)}}=(\mathbf{G}^{(10)},\mathbf{G}^{(1c)},\mathbf{G}^{(1s)},\mathbf{G}^{(20)},\mathbf{G}^{(2c)},\mathbf{G}^{(2s)})^{\mathrm{T}} \tag{5.25}$$

$$\begin{aligned}
\mathbf{G}^{(i0)} &= (G_0^{(i0)},G_1^{(i0)},\cdots,G_{4N+1}^{(i0)}),\\
\mathbf{G}^{(ic)} &= (\mathbf{G}_1^{(ic)},\mathbf{G}_2^{(ic)},\cdots,\mathbf{G}_N^{(ic)})^{\mathrm{T}},\\
\mathbf{G}^{(is)} &= (\mathbf{G}_1^{(is)},\mathbf{G}_2^{(is)},\cdots,\mathbf{G}_N^{(is)})^{\mathrm{T}}
\end{aligned} \tag{5.26}$$

for $i=1,2;$ and $N=1,2,\cdots\infty$ with

$$\begin{aligned}
\mathbf{G}_k^{(ic)} &= (G_{k0}^{(ic)},G_{k1}^{(ic)},\cdots,G_{k(4N+1)}^{(ic)}),\\
\mathbf{G}_k^{(is)} &= (G_{k0}^{(is)},G_{k1}^{(is)},\cdots,G_{k(4N+1)}^{(is)})
\end{aligned} \tag{5.27}$$

for $k=1,2,\cdots N.$ The corresponding components are

$$\begin{aligned}
G_r^{(10)} &= -(\alpha_{15}+\alpha_{16}\frac{\partial f_{11}^{16}}{\partial a_{10}})\delta_0^r-\delta_{2N+1}^r(\alpha_3\frac{\partial f_1^3}{\partial a_{20}}+\alpha_5\frac{\partial f_1^5}{\partial a_{20}}+\alpha_6\frac{\partial f_1^6}{\partial a_{20}}+\alpha_9\frac{\partial f_1^9}{\partial a_{20}}+\\
&\quad \alpha_{10}\frac{\partial f_1^{10}}{\partial a_{20}}+\alpha_{12}\frac{\partial f_1^{12}}{\partial a_{20}}+\alpha_{13})-g_r^{(10)}\\
G_{kr}^{(1c)} &= -\alpha_{16}\frac{\partial f_{21}^{16}}{\partial a_{10}}\delta_0^r-\delta_{2N+1}^r(\alpha_3\frac{\partial f_2^3}{\partial a_{20}}+\alpha_5\frac{\partial f_2^5}{\partial a_{20}}+\alpha_6\frac{\partial f_2^6}{\partial a_{20}}+\alpha_9\frac{\partial f_2^9}{\partial a_{20}}\\
&\quad +\alpha_{10}\frac{\partial f_2^{10}}{\partial a_{20}}+\alpha_{12}\frac{\partial f_2^{12}}{\partial a_{20}})-g_{kr}^{(1c)}\\
G_{kr}^{(1s)} &= -\alpha_{16}\frac{\partial f_{31}^{16}}{\partial a_{10}}\delta_0^r-\delta_{2N+1}^r(\alpha_3\frac{\partial f_3^3}{\partial a_{20}}+\alpha_5\frac{\partial f_3^5}{\partial a_{20}}+\alpha_6\frac{\partial f_3^6}{\partial a_{20}}+\alpha_9\frac{\partial f_3^9}{\partial a_{20}}+\alpha_{10}\frac{\partial f_3^{10}}{\partial a_{20}}\\
&\quad +\alpha_{12}\frac{\partial f_3^{12}}{\partial a_{20}})-g_{kr}^{(1s)}\\
G_r^{(20)} &= -\delta_{2N+1}^r(\beta_3\frac{\partial f_1^3}{\partial a_{20}}+\beta_5\frac{\partial f_1^5}{\partial a_{20}}+\beta_6\frac{\partial f_1^6}{\partial a_{20}}+\beta_9\frac{\partial f_1^9}{\partial a_{20}}+\beta_{10}\frac{\partial f_1^{10}}{\partial a_{20}}+\beta_{12}\frac{\partial f_1^{12}}{\partial a_{20}}+\beta_{13}\\
&\quad +\beta_{15}+\beta_{16}\frac{\partial f_{12}^{16}}{\partial a_{20}})-g_r^{(20)}\\
G_{kr}^{(2c)} &= -\delta_{2N+1}^r(\beta_3\frac{\partial f_2^3}{\partial a_{20}}+\beta_5\frac{\partial f_2^5}{\partial a_{20}}+\beta_6\frac{\partial f_2^6}{\partial a_{20}}+\beta_9\frac{\partial f_2^9}{\partial a_{20}}+\beta_{10}\frac{\partial f_2^{10}}{\partial a_{20}}+\beta_{12}\frac{\partial f_2^{12}}{\partial a_{20}}+
\end{aligned} \tag{5.28}$$

$$\beta_{16}\frac{\partial f_{22}^{16}}{\partial a_{20}}) - g_{kr}^{(2c)}$$

$$G_{kr}^{(2s)} = -\delta_{2N+1}^{r}(\beta_3\frac{\partial f_3^3}{\partial a_{20}} + \beta_5\frac{\partial f_3^5}{\partial a_{20}} + \beta_6\frac{\partial f_3^6}{\partial a_{20}} + \beta_9\frac{\partial f_3^9}{\partial a_{20}} + \beta_{10}\frac{\partial f_3^{10}}{\partial a_{20}} + \beta_{12}\frac{\partial f_3^{12}}{\partial a_{20}} +$$

$$\beta_{16}\frac{\partial f_{32}^{16}}{\partial a_{20}}) - g_{kr}^{(2s)}$$

where for $r = 0,1,\cdots,4N+1$.

The derivative of the constant term for the transverse motion is

$$g_r^{(10)} = g_{r1}^{(10)} + g_{r2}^{(10)} + g_{r3}^{(10)} + g_{r4}^{(10)}, \tag{5.29}$$

with

$$g_{r1}^{(10)} = \sum_{n=1}^{N}\delta_n^r\frac{\partial Q_n}{\partial b_{1n}}(\alpha_2\frac{\partial f_1^2}{\partial Q_n} + \alpha_4\frac{\partial f_1^4}{\partial Q_n} + \alpha_5\frac{\partial f_1^5}{\partial Q_n} + \alpha_8\frac{\partial f_1^8}{\partial Q_n} + \alpha_9\frac{\partial f_1^9}{\partial Q_n}$$
$$+\alpha_{10}\frac{\partial f_1^{10}}{\partial Q_n} + \alpha_{16}\frac{\partial f_{11}^{16}}{\partial b_{1n}})$$
$$g_{r2}^{(10)} = \sum_{n=1}^{N}\delta_{n+N}^r\frac{\partial P_n}{\partial c_{1n}}(\alpha_2\frac{\partial f_1^2}{\partial P_n} + \alpha_4\frac{\partial f_1^4}{\partial P_n} + \alpha_5\frac{\partial f_1^5}{\partial P_n} + \alpha_8\frac{\partial f_1^8}{\partial P_n} + \alpha_9\frac{\partial f_1^9}{\partial P_n}$$
$$+\alpha_{10}\frac{\partial f_1^{10}}{\partial P_n} + \alpha_{16}\frac{\partial f_{11}^{16}}{\partial c_{1n}}) \tag{5.30}$$
$$g_{r3}^{(10)} = \sum_{n=1}^{N}\delta_{n+2N+1}^r(\frac{\partial C_n}{\partial b_{2n}}(\alpha_1\frac{\partial f_1^1}{\partial C_n} + \alpha_2\frac{\partial f_1^2}{\partial C_n} + \alpha_3\frac{\partial f_1^3}{\partial C_n} + \alpha_4\frac{\partial f_1^4}{\partial C_n} + \alpha_5\frac{\partial f_1^5}{\partial C_n}$$
$$+\alpha_6\frac{\partial f_1^6}{\partial C_n}) + \alpha_3\frac{\partial f_1^3}{\partial b_{2n}} + \alpha_5\frac{\partial f_1^5}{\partial b_{2n}} + \alpha_6\frac{\partial f_1^6}{\partial b_{2n}} + \alpha_9\frac{\partial f_1^9}{\partial b_{2n}} + \alpha_{10}\frac{\partial f_1^{10}}{\partial b_{2n}}$$
$$+\alpha_{12}\frac{\partial f_1^{12}}{\partial b_{2n}})$$
$$g_{r4}^{(10)} = \sum_{n=1}^{N}\delta_{n+3N+1}^r(\frac{\partial B_n}{\partial c_{2n}}(\alpha_1\frac{\partial f_1^1}{\partial B_n} + \alpha_2\frac{\partial f_1^2}{\partial B_n} + \alpha_3\frac{\partial f_1^3}{\partial B_n} + \alpha_4\frac{\partial f_1^4}{\partial B_n} + \alpha_5\frac{\partial f_1^5}{\partial B_n}$$

$$+\alpha_6 \frac{\partial f_1^6}{\partial B_n}) + \alpha_3 \frac{\partial f_1^3}{\partial c_{2n}} + \alpha_5 \frac{\partial f_1^5}{\partial c_{2n}} + \alpha_6 \frac{\partial f_1^6}{\partial c_{2n}} + \alpha_9 \frac{\partial f_1^9}{\partial c_{2n}} + \alpha_{10} \frac{\partial f_1^{10}}{\partial c_{2n}}$$
$$+\alpha_{12} \frac{\partial f_1^{12}}{\partial c_{2n}})$$

The derivative related to the cosine term for the transverse motion is

$$g_{kr}^{(1c)} = g_{kr1}^{(1c)} + g_{kr2}^{(1c)} + g_{kr3}^{(1c)} + g_{kr4}^{(1c)} \tag{5.31}$$

with

$$\begin{aligned}
g_{kr1}^{(1c)} &= \sum_{n=1}^{N} \delta_n^r (\frac{\partial Q_n}{\partial b_{1n}}(\alpha_2 \frac{\partial f_2^2}{\partial Q_n} + \alpha_4 \frac{\partial f_2^4}{\partial Q_n} + \alpha_5 \frac{\partial f_2^5}{\partial Q_n} + \alpha_8 \frac{\partial f_2^8}{\partial Q_n} + \alpha_9 \frac{\partial f_2^9}{\partial Q_n} + \\
&\quad \alpha_{10} \frac{\partial f_2^{10}}{\partial Q_n}) + \alpha_{16} \frac{\partial f_{21}^{16}}{\partial b_{1n}} + \delta_n^k [\alpha_{15} - (k\Omega)^2])\\
g_{kr2}^{(1c)} &= \sum_{n=1}^{N} \delta_{n+N}^r (\frac{\partial P_n}{\partial c_{1n}}(\alpha_2 \frac{\partial f_2^2}{\partial P_n} + \alpha_4 \frac{\partial f_2^4}{\partial P_n} + \alpha_5 \frac{\partial f_2^5}{\partial P_n} + \alpha_8 \frac{\partial f_2^8}{\partial P_n} + \alpha_9 \frac{\partial f_2^9}{\partial P_n} + \\
&\quad \alpha_{10} \frac{\partial f_2^{10}}{\partial P_n}) + \alpha_{16} \frac{\partial f_{21}^{16}}{\partial c_{1n}} + \delta_n^k k\Omega[\alpha_{11} + \alpha_{14}])\\
g_{kr3}^{(1c)} &= \sum_{n=1}^{N} \delta_{n+2N+1}^r (\frac{\partial C_n}{\partial b_{2n}}(\alpha_1 \frac{\partial f_2^1}{\partial C_n} + \alpha_2 \frac{\partial f_2^2}{\partial C_n} + \alpha_3 \frac{\partial f_2^3}{\partial C_n} + \alpha_4 \frac{\partial f_2^4}{\partial C_n} + \alpha_5 \frac{\partial f_2^5}{\partial C_n} + \\
&\quad \alpha_6 \frac{\partial f_2^6}{\partial C_n}) + \alpha_3 \frac{\partial f_2^3}{\partial b_{2n}} + \alpha_5 \frac{\partial f_2^5}{\partial b_{2n}} + \alpha_6 \frac{\partial f_2^6}{\partial b_{2n}} + \alpha_9 \frac{\partial f_2^9}{\partial b_{2n}} + \alpha_{10} \frac{\partial f_2^{10}}{\partial b_{2n}} + \alpha_{12} \frac{\partial f_2^{12}}{\partial b_{2n}}\\
&\quad +\alpha_{13} \frac{\partial f_2^{13}}{\partial b_{2n}})\\
g_{kr4}^{(1c)} &= \sum_{n=1}^{N} \delta_{n+3N+1}^r (\frac{\partial B_n}{\partial c_{2n}}(\alpha_1 \frac{\partial f_2^1}{\partial B_n} + \alpha_2 \frac{\partial f_2^2}{\partial B_n} + \alpha_3 \frac{\partial f_2^3}{\partial B_n} + \alpha_4 \frac{\partial f_2^4}{\partial B_n} + \alpha_5 \frac{\partial f_2^5}{\partial B_n} + \\
&\quad \alpha_6 \frac{\partial f_2^6}{\partial B_n}) + \alpha_3 \frac{\partial f_2^3}{\partial c_{2n}} + \alpha_5 \frac{\partial f_2^5}{\partial c_{2n}} + \alpha_6 \frac{\partial f_2^6}{\partial c_{2n}} + \alpha_7 \frac{\partial f_2^7}{\partial c_{2n}} + \alpha_9 \frac{\partial f_2^9}{\partial c_{2n}} + \alpha_{10} \frac{\partial f_2^{10}}{\partial c_{2n}}\\
&\quad +\alpha_{12} \frac{\partial f_2^{12}}{\partial c_{2n}})
\end{aligned} \tag{5.32}$$

The derivative related to the sine term for the transverse motion is

$$g_{kr}^{(1s)} = g_{kr1}^{(1s)} + g_{kr2}^{(1s)} + g_{kr3}^{(1s)} + g_{kr4}^{(1s)} \tag{5.33}$$

with

$$\begin{aligned}
g_{kr1}^{(1s)} &= \sum_{n=1}^{N} \delta_n^r \Big(\frac{\partial Q_n}{\partial b_{1n}}(\alpha_2 \frac{\partial f_3^2}{\partial Q_n} + \alpha_4 \frac{\partial f_3^4}{\partial Q_n} + \alpha_5 \frac{\partial f_3^5}{\partial Q_n} + \alpha_8 \frac{\partial f_3^8}{\partial Q_n} + \alpha_9 \frac{\partial f_3^9}{\partial Q_n} + \alpha_{10} \frac{\partial f_3^{10}}{\partial Q_n}) \\
&\quad + \alpha_{16} \frac{\partial f_{31}^{16}}{\partial b_{1n}} + \delta_n^k [-(\alpha_{11} + \alpha_{14})(k\Omega)]\Big) \\
g_{kr2}^{(1s)} &= \sum_{n=1}^{N} \delta_{n+N}^r \Big(\frac{\partial P_n}{\partial c_{1n}}(\alpha_2 \frac{\partial f_3^2}{\partial P_n} + \alpha_4 \frac{\partial f_3^4}{\partial P_n} + \alpha_5 \frac{\partial f_3^5}{\partial P_n} + \alpha_8 \frac{\partial f_3^8}{\partial P_n} + \alpha_9 \frac{\partial f_3^9}{\partial P_n} \alpha_{10} \frac{\partial f_3^{10}}{\partial P_n}) \\
&\quad + \alpha_{16} \frac{\partial f_{31}^{16}}{\partial c_{1n}} + \delta_n^k [\alpha_{15} - (k\Omega)^2]\Big) \\
g_{kr3}^{(1s)} &= \sum_{n=1}^{N} \delta_{n+2N+1}^r \Big(\frac{\partial C_n}{\partial b_{2n}}(\alpha_1 \frac{\partial f_3^1}{\partial C_n} + \alpha_2 \frac{\partial f_3^2}{\partial C_n} + \alpha_3 \frac{\partial f_3^3}{\partial C_n} + \alpha_4 \frac{\partial f_3^4}{\partial C_n} + \alpha_5 \frac{\partial f_3^5}{\partial C_n} \\
&\quad + \alpha_6 \frac{\partial f_3^6}{\partial C_n}) + \alpha_3 \frac{\partial f_3^3}{\partial b_{2n}} + \alpha_5 \frac{\partial f_3^5}{\partial b_{2n}} + \alpha_6 \frac{\partial f_3^6}{\partial b_{2n}} + \alpha_7 \frac{\partial f_3^7}{\partial b_{2n}} + \alpha_9 \frac{\partial f_3^9}{\partial b_{2n}} + \alpha_{10} \frac{\partial f_3^{10}}{\partial b_{2n}} \\
&\quad + \alpha_{12} \frac{\partial f_3^{12}}{\partial b_{2n}}\Big) \\
g_{kr4}^{(1s)} &= \sum_{n=1}^{N} \delta_{n+3N+1}^r \Big(\frac{\partial B_n}{\partial c_{2n}}(\alpha_1 \frac{\partial f_3^1}{\partial B_n} + \alpha_2 \frac{\partial f_3^2}{\partial B_n} + \alpha_3 \frac{\partial f_3^3}{\partial B_n} + \alpha_4 \frac{\partial f_3^4}{\partial B_n} + \alpha_5 \frac{\partial f_3^5}{\partial B_n} \\
&\quad + \alpha_6 \frac{\partial f_3^6}{\partial B_n}) + \alpha_3 \frac{\partial f_3^3}{\partial c_{2n}} + \alpha_5 \frac{\partial f_3^5}{\partial c_{2n}} + \alpha_6 \frac{\partial f_3^6}{\partial c_{2n}} + \alpha_9 \frac{\partial f_3^9}{\partial c_{2n}} + \alpha_{10} \frac{\partial f_3^{10}}{\partial c_{2n}} + \\
&\quad \alpha_{12} \frac{\partial f_3^{12}}{\partial c_{2n}} + \alpha_{13} \frac{\partial f_3^{13}}{\partial c_{2n}}\Big)
\end{aligned} \tag{5.34}$$

The derivative relatives to the constant for the torsional motion are

$$g_r^{(20)} = g_{r1}^{(20)} + g_{r2}^{(20)} + g_{r3}^{(20)} + g_{r4}^{(20)} \tag{5.35}$$

with

$$
\begin{aligned}
g_{r1}^{(20)} &= \sum_{n=1}^{N} \delta_n^r \frac{\partial Q_n}{\partial b_{1n}} \left(\beta_2 \frac{\partial f_1^2}{\partial Q_n} + \beta_4 \frac{\partial f_1^4}{\partial Q_n} + \beta_5 \frac{\partial f_1^5}{\partial Q_n} + \beta_8 \frac{\partial f_1^8}{\partial Q_n} + \beta_9 \frac{\partial f_1^9}{\partial Q_n} \right.\\
&\quad \left. + \beta_{10} \frac{\partial f_1^{10}}{\partial Q_n}\right) \\
g_{r2}^{(20)} &= \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial P_n}{\partial c_{1n}} \left(\beta_2 \frac{\partial f_1^2}{\partial P_n} + \beta_4 \frac{\partial f_1^4}{\partial P_n} + \beta_5 \frac{\partial f_1^5}{\partial P_n} + \beta_8 \frac{\partial f_1^8}{\partial P_n} + \beta_9 \frac{\partial f_1^9}{\partial P_n} \right.\\
&\quad \left. + \beta_{10} \frac{\partial f_1^{10}}{\partial P_n}\right) \\
g_{r3}^{(20)} &= \sum_{n=1}^{N} \delta_{n+2N+1}^r \left(\frac{\partial C_n}{\partial b_{2n}} \left(\beta_1 \frac{\partial f_1^1}{\partial C_n} + \beta_2 \frac{\partial f_1^2}{\partial C_n} + \beta_3 \frac{\partial f_1^3}{\partial C_n} + \beta_4 \frac{\partial f_1^4}{\partial C_n} + \beta_5 \frac{\partial f_1^5}{\partial C_n} \right.\right.\\
&\quad \left. + \beta_6 \frac{\partial f_1^6}{\partial C_n}\right) + \beta_3 \frac{\partial f_1^3}{\partial b_{2n}} + \beta_5 \frac{\partial f_1^5}{\partial b_{2n}} + \beta_6 \frac{\partial f_1^6}{\partial b_{2n}} + \beta_9 \frac{\partial f_1^9}{\partial b_{2n}} + \beta_{10} \frac{\partial f_1^{10}}{\partial b_{2n}} \\
&\quad \left. + \beta_{12} \frac{\partial f_1^{12}}{\partial b_{2n}} + \beta_{16} \frac{\partial f_{12}^{16}}{\partial b_{2n}}\right) \\
g_{r4}^{(20)} &= \sum_{n=1}^{N} \delta_{n+3N+1}^r \left(\frac{\partial B_n}{\partial c_{2n}} \left(\beta_1 \frac{\partial f_1^1}{\partial B_n} + \beta_2 \frac{\partial f_1^2}{\partial B_n} + \beta_3 \frac{\partial f_1^3}{\partial B_n} + \beta_4 \frac{\partial f_1^4}{\partial B_n} + \beta_5 \frac{\partial f_1^5}{\partial B_n} \right.\right.\\
&\quad \left. + \beta_6 \frac{\partial f_1^6}{\partial B_n}\right) + \beta_3 \frac{\partial f_1^3}{\partial c_{2n}} + \beta_5 \frac{\partial f_1^5}{\partial c_{2n}} + \beta_6 \frac{\partial f_1^6}{\partial c_{2n}} + \beta_9 \frac{\partial f_1^9}{\partial c_{2n}} + \beta_{10} \frac{\partial f_1^{10}}{\partial c_{2n}} \\
&\quad \left. + \beta_{12} \frac{\partial f_1^{12}}{\partial c_{2n}} + \beta_{16} \frac{\partial f_{12}^{16}}{\partial c_{2n}}\right)
\end{aligned} \tag{5.36}
$$

The derivative related to the cosine term for the torsional motion is

$$g_{kr}^{(2c)} = g_{kr1}^{(2c)} + g_{kr2}^{(2c)} + g_{kr3}^{(2c)} + g_{kr4}^{(2c)} \tag{5.37}$$

where

$$
\begin{aligned}
g_{kr1}^{(2c)} &= \sum_{n=1}^{N} \delta_n^r \frac{\partial Q_n}{\partial b_{1n}} \left(\beta_2 \frac{\partial f_2^2}{\partial Q_n} + \beta_4 \frac{\partial f_2^4}{\partial Q_n} + \beta_5 \frac{\partial f_2^5}{\partial Q_n} + \beta_8 \frac{\partial f_2^8}{\partial Q_n} + \beta_9 \frac{\partial f_2^9}{\partial Q_n} \right.\\
&\quad \left. + \beta_{10} \frac{\partial f_2^{10}}{\partial Q_n}\right) \\
g_{kr2}^{(2c)} &= \sum_{n=1}^{N} \delta_{n+N}^r \left(\frac{\partial P_i}{\partial c_{1n}} \left(\beta_2 \frac{\partial f_2^2}{\partial P_n} + \beta_4 \frac{\partial f_2^4}{\partial P_n} + \beta_5 \frac{\partial f_2^5}{\partial P_n} + \beta_8 \frac{\partial f_2^8}{\partial P_n} + \beta_9 \frac{\partial f_2^9}{\partial P_n} \right.\right.\\
&\quad \left.\left. + \beta_{10} \frac{\partial f_2^{10}}{\partial P_n}\right) + \delta_n^k \beta_{11}\right)
\end{aligned}
$$

$$g_{kr3}^{(2c)}=\sum_{n=1}^{N}\delta_{n+2N+1}^{r}(\frac{\partial C_n}{\partial b_{2n}}(\beta_1\frac{\partial f_2^1}{\partial C_n}+\beta_2\frac{\partial f_2^2}{\partial C_n}+\beta_3\frac{\partial f_2^3}{\partial C_n}+\beta_4\frac{\partial f_2^4}{\partial C_n}+\beta_5\frac{\partial f_2^5}{\partial C_n}$$
$$+\beta_6\frac{\partial f_2^6}{\partial C_n})+\beta_3\frac{\partial f_2^3}{\partial b_{2n}}+\beta_5\frac{\partial f_2^5}{\partial b_{2n}}+\beta_6\frac{\partial f_2^6}{\partial b_{2n}}+\beta_9\frac{\partial f_2^9}{\partial b_{2n}}+\beta_{10}\frac{\partial f_2^{10}}{\partial b_{2n}}+ \tag{5.38}$$
$$\beta_{12}\frac{\partial f_2^{12}}{\partial b_{2n}}+\beta_{13}\frac{\partial f_2^{13}}{\partial b_{2n}}+\beta_{15}\frac{\partial f_2^{15}}{\partial b_{2n}}+\beta_{16}\frac{\partial f_{22}^{16}}{\partial b_{2n}}-\delta_n^k(k\Omega)^2)$$
$$g_{kr4}^{(2c)}=\sum_{n=1}^{N}\delta_{n+3N+1}^{r}(\frac{\partial B_n}{\partial c_{2n}}(\beta_1\frac{\partial f_2^1}{\partial B_n}+\beta_2\frac{\partial f_2^2}{\partial B_n}+\beta_3\frac{\partial f_2^3}{\partial B_n}+\beta_4\frac{\partial f_2^4}{\partial B_n}+\beta_5\frac{\partial f_2^5}{\partial B_n}$$
$$+\beta_6\frac{\partial f_2^6}{\partial B_n})+\beta_3\frac{\partial f_2^3}{\partial c_{2n}}+\beta_5\frac{\partial f_2^5}{\partial c_{2n}}+\beta_6\frac{\partial f_2^6}{\partial c_{2n}}+\beta_9\frac{\partial f_2^9}{\partial c_{2n}}+\beta_{10}\frac{\partial f_2^{10}}{\partial c_{2n}}+$$
$$\beta_{16}\frac{\partial f_{22}^{16}}{\partial c_{2n}}+\beta_{12}\frac{\partial f_2^{12}}{\partial c_{2n}}+\beta_{12}\frac{\partial f_2^{14}}{\partial c_{2n}})$$

The derivative related to the sine term for the torsional motion is

$$g_{kr}^{(2s)}=g_{kr1}^{(2s)}+g_{kr2}^{(2s)}+g_{kr3}^{(2s)}+g_{kr4}^{(2s)} \tag{5.39}$$

Where

$$g_{kr1}^{(2s)}=\sum_{n=1}^{N}\delta_{n}^{r}(\frac{\partial Q_n}{\partial b_{1n}}(\beta_2\frac{\partial f_3^2}{\partial Q_n}+\beta_4\frac{\partial f_3^4}{\partial Q_n}+\beta_5\frac{\partial f_3^5}{\partial Q_n}+\beta_8\frac{\partial f_3^8}{\partial Q_n}+\beta_9\frac{\partial f_3^9}{\partial Q_n}$$
$$+\beta_{10}\frac{\partial f_3^{10}}{\partial Q_n})-\delta_n^k\beta_{11}k\Omega)$$
$$g_{kr2}^{(2s)}=\sum_{n=1}^{N}\delta_{n+N}^{r}\frac{\partial P_n}{\partial c_{1n}}(\beta_2\frac{\partial f_3^2}{\partial P_n}+\beta_4\frac{\partial f_3^4}{\partial P_n}+\beta_5\frac{\partial f_3^5}{\partial P_n}+\beta_8\frac{\partial f_3^8}{\partial P_n}+\beta_9\frac{\partial f_3^9}{\partial P_n}$$
$$+\beta_{10}\frac{\partial f_3^{10}}{\partial P_n}) \tag{5.40}$$
$$g_{kr3}^{(2s)}=\sum_{n=1}^{N}\delta_{n+2N+1}^{r}(\frac{\partial C_n}{\partial b_{2n}}(\beta_1\frac{\partial f_3^1}{\partial C_n}+\beta_2\frac{\partial f_3^2}{\partial C_n}+\beta_3\frac{\partial f_3^3}{\partial C_n}+\beta_4\frac{\partial f_3^4}{\partial C_n}+\beta_5\frac{\partial f_3^5}{\partial C_n}$$
$$+\beta_6\frac{\partial f_3^6}{\partial C_n})+\beta_3\frac{\partial f_3^3}{\partial b_{2n}}+\beta_5\frac{\partial f_3^5}{\partial b_{2n}}+\beta_6\frac{\partial f_3^6}{\partial b_{2n}}+\beta_7\frac{\partial f_3^7}{\partial b_{2n}}+\beta_9\frac{\partial f_3^9}{\partial b_{2n}}+$$
$$\beta_{10}\frac{\partial f_3^{10}}{\partial b_{2n}}+\beta_{12}\frac{\partial f_3^{12}}{\partial b_{2n}}+\beta_{14}\frac{\partial f_3^{14}}{\partial b_{2n}}+\beta_{16}\frac{\partial f_{32}^{16}}{\partial b_{2n}})$$

$$g_{kr4}^{(2s)} = \sum_{n=1}^{N} \delta_{n+3N+1}^{r} (\frac{\partial B_n}{\partial c_{2n}} (\beta_1 \frac{\partial f_3^1}{\partial B_n} + \beta_2 \frac{\partial f_3^2}{\partial B_n} + \beta_3 \frac{\partial f_3^3}{\partial B_n} + \beta_4 \frac{\partial f_3^4}{\partial B_n} + \beta_5 \frac{\partial f_3^5}{\partial B_n}$$
$$+\beta_6 \frac{\partial f_3^6}{\partial B_n}) + \beta_3 \frac{\partial f_3^3}{\partial c_{2n}} + \beta_5 \frac{\partial f_3^5}{\partial c_{2n}} + \beta_6 \frac{\partial f_3^6}{\partial c_{2n}} + \beta_9 \frac{\partial f_3^9}{\partial c_{2n}} + \beta_{10} \frac{\partial f_3^{10}}{\partial c_{2n}} +$$
$$\beta_{12} \frac{\partial f_3^{12}}{\partial c_{2n}} + \beta_{13} \frac{\partial f_3^{13}}{\partial c_{2n}} + \beta_{15} \frac{\partial f_3^{15}}{\partial c_{2n}} + \beta_{16} \frac{\partial f_{32}^{16}}{\partial c_{2n}} - \delta_n^k (k\Omega)^2)$$

The H-matrix is

$$\mathbf{H} = \frac{\partial \mathbf{g}^{(m)}}{\partial \mathbf{z}_1^{(m)}} = (\mathbf{H}^{(10)}, \mathbf{H}^{(1c)}, \mathbf{H}^{(1s)}, \mathbf{H}^{(20)}, \mathbf{H}^{(2c)}, \mathbf{H}^{(2s)})^{\mathrm{T}} \tag{5.41}$$

where

$$\begin{aligned} \mathbf{H}^{(i0)} &= (H_0^{(i0)}, H_1^{(i0)}, \cdots, H_{4N+1}^{(i0)}), \\ \mathbf{H}^{(ic)} &= (\mathbf{H}_1^{(ic)}, \mathbf{H}_2^{(ic)}, \cdots, \mathbf{H}_N^{(ic)})^{\mathrm{T}}, \\ \mathbf{H}^{(is)} &= (\mathbf{H}_1^{(is)}, \mathbf{H}_2^{(is)}, \cdots, \mathbf{H}_N^{(is)})^{\mathrm{T}} \end{aligned} \tag{5.42}$$

for $i = 1, 2$ and $N = 1, 2, \cdots \infty,$ with

$$\begin{aligned} \mathbf{H}_k^{(ic)} &= (H_{k0}^{(ic)}, H_{k1}^{(ic)}, \cdots, H_{k(4N+1)}^{(ic)}), \\ \mathbf{H}_k^{(is)} &= (H_{k0}^{(is)}, H_{k1}^{(is)}, \cdots, H_{k(4N+1)}^{(is)}) \end{aligned} \tag{5.43}$$

for $k = 1, 2, \cdots N.$ The corresponding components are

$$H_r^{(10)} = -\delta_0^r (\alpha_2 \frac{\partial f_1^2}{\partial \dot{a}_{10}} + \alpha_4 \frac{\partial f_1^4}{\partial \dot{a}_{10}} + \alpha_5 \frac{\partial f_1^5}{\partial \dot{a}_{10}} + \alpha_8 \frac{\partial f_1^8}{\partial \dot{a}_{10}} + \alpha_9 \frac{\partial f_1^9}{\partial \dot{a}_{10}} + \alpha_{10} \frac{\partial f_1^{10}}{\partial \dot{a}_{10}}$$
$$+\alpha_{11} + \alpha_{14}) - \delta_{2N+1}^r (\alpha_1 \frac{\partial f_1^1}{\partial \dot{a}_{20}} + \alpha_2 \frac{\partial f_1^2}{\partial \dot{a}_{20}} + \alpha_3 \frac{\partial f_1^3}{\partial \dot{a}_{20}} + \alpha_4 \frac{\partial f_1^4}{\partial \dot{a}_{20}} + \alpha_5 \frac{\partial f_1^5}{\partial \dot{a}_{20}}$$
$$+\alpha_6 \frac{\partial f_1^6}{\partial \dot{a}_{20}} + \alpha_7) - Z_r^{(10)},$$

$$
\begin{aligned}
H_{kr}^{(1c)} &= -\delta_0^r(\alpha_2 \frac{\partial f_2^2}{\partial \dot{a}_{10}} + \alpha_4 \frac{\partial f_2^4}{\partial \dot{a}_{10}} + \alpha_5 \frac{\partial f_2^5}{\partial \dot{a}_{10}} + \alpha_8 \frac{\partial f_2^8}{\partial \dot{a}_{10}} + \alpha_9 \frac{\partial f_2^9}{\partial \dot{a}_{10}} + \alpha_{10} \frac{\partial f_2^{10}}{\partial \dot{a}_{10}}) \\
&\quad -\delta_{2N+1}^r(\alpha_1 \frac{\partial f_2^1}{\partial \dot{a}_{20}} + \alpha_2 \frac{\partial f_2^2}{\partial \dot{a}_{20}} + \alpha_3 \frac{\partial f_2^3}{\partial \dot{a}_{20}} + \alpha_4 \frac{\partial f_2^4}{\partial \dot{a}_{20}} + \alpha_5 \frac{\partial f_2^5}{\partial \dot{a}_{20}} + \alpha_6 \frac{\partial f_2^6}{\partial \dot{a}_{20}} \\
&\quad +\alpha_9 \frac{\partial f_2^9}{\partial \dot{a}_{20}}) - Z_{kr}^{(1c)}, \\
H_{kr}^{(1s)} &= -\delta_0^r(\alpha_2 \frac{\partial f_3^2}{\partial \dot{a}_{10}} + \alpha_4 \frac{\partial f_3^4}{\partial \dot{a}_{10}} + \alpha_5 \frac{\partial f_3^5}{\partial \dot{a}_{10}} + \alpha_8 \frac{\partial f_3^8}{\partial \dot{a}_{10}} + \alpha_9 \frac{\partial f_3^9}{\partial \dot{a}_{10}} + \alpha_{10} \frac{\partial f_3^{10}}{\partial \dot{a}_{10}}) \\
&\quad -\delta_{2N+1}^r(\alpha_1 \frac{\partial f_3^1}{\partial \dot{a}_{20}} + \alpha_2 \frac{\partial f_3^2}{\partial \dot{a}_{20}} + \alpha_3 \frac{\partial f_3^3}{\partial \dot{a}_{20}} + \alpha_4 \frac{\partial f_3^4}{\partial \dot{a}_{20}} + \alpha_5 \frac{\partial f_3^5}{\partial \dot{a}_{20}} + \alpha_6 \frac{\partial f_3^6}{\partial \dot{a}_{20}}) \\
&\quad -Z_{kr}^{(1s)},
\end{aligned} \tag{5.44}
$$

$$
\begin{aligned}
H_r^{(20)} &= -\delta_0^r(\beta_2 \frac{\partial f_1^2}{\partial \dot{a}_{10}} + \beta_4 \frac{\partial f_1^4}{\partial \dot{a}_{10}} + \beta_5 \frac{\partial f_1^5}{\partial \dot{a}_{10}} + \beta_8 \frac{\partial f_1^8}{\partial \dot{a}_{10}} + \beta_9 \frac{\partial f_1^9}{\partial \dot{a}_{10}} + \beta_{10} \frac{\partial f_1^{10}}{\partial \dot{a}_{10}} \\
&\quad +\alpha_{11}) - \delta_{2N+1}^r(\beta_1 \frac{\partial f_1^1}{\partial \dot{a}_{20}} + \beta_2 \frac{\partial f_1^2}{\partial \dot{a}_{20}} + \beta_3 \frac{\partial f_1^3}{\partial \dot{a}_{20}} + \beta_4 \frac{\partial f_1^4}{\partial \dot{a}_{20}} + \beta_5 \frac{\partial f_1^5}{\partial \dot{a}_{20}} \\
&\quad +\beta_6 \frac{\partial f_1^6}{\partial \dot{a}_{20}} + \beta_7 + \beta_{14}) - Z_r^{(20)}, \\
H_{kr}^{(2c)} &= -\delta_0^r(\beta_2 \frac{\partial f_2^2}{\partial \dot{a}_{10}} + \beta_4 \frac{\partial f_2^4}{\partial \dot{a}_{10}} + \beta_5 \frac{\partial f_2^5}{\partial \dot{a}_{10}} + \beta_8 \frac{\partial f_2^8}{\partial \dot{a}_{10}} + \beta_9 \frac{\partial f_2^9}{\partial \dot{a}_{10}} + \beta_{10} \frac{\partial f_2^{10}}{\partial \dot{a}_{10}}) \\
&\quad -\delta_{2N+1}^r(\beta_1 \frac{\partial f_2^1}{\partial \dot{a}_{20}} + \beta_2 \frac{\partial f_2^2}{\partial \dot{a}_{20}} + \beta_3 \frac{\partial f_2^3}{\partial \dot{a}_{20}} + \beta_4 \frac{\partial f_2^4}{\partial \dot{a}_{20}} + \beta_5 \frac{\partial f_2^5}{\partial \dot{a}_{20}} + \beta_6 \frac{\partial f_2^6}{\partial \dot{a}_{20}} \\
&\quad +\beta_9 \frac{\partial f_2^9}{\partial \dot{a}_{20}}) - Z_{kr}^{(2c)}, \\
H_{kr}^{(2s)} &= -\delta_0^r(\beta_2 \frac{\partial f_3^2}{\partial \dot{a}_{10}} + \beta_4 \frac{\partial f_3^4}{\partial \dot{a}_{10}} + \beta_5 \frac{\partial f_3^5}{\partial \dot{a}_{10}} + \beta_8 \frac{\partial f_3^8}{\partial \dot{a}_{10}} + \beta_9 \frac{\partial f_3^9}{\partial \dot{a}_{10}} + \beta_{10} \frac{\partial f_3^{10}}{\partial \dot{a}_{10}}) \\
&\quad -\delta_{2N+1}^r(\beta_1 \frac{\partial f_3^1}{\partial \dot{a}_{20}} + \beta_2 \frac{\partial f_3^2}{\partial \dot{a}_{20}} + \beta_3 \frac{\partial f_3^3}{\partial \dot{a}_{20}} + \beta_4 \frac{\partial f_3^4}{\partial \dot{a}_{20}} + \beta_5 \frac{\partial f_3^5}{\partial \dot{a}_{20}} + \beta_6 \frac{\partial f_3^6}{\partial \dot{a}_{20}}) \\
&\quad -Z_{kr}^{(2s)}
\end{aligned} \tag{5.45}
$$

for $r = 0, 1, \cdots, 4N+1$.

The derivative of the constant term for the transverse motion is

$$
Z_r^{(10)} = Z_{r1}^{(10)} + Z_{r2}^{(10)} + Z_{r3}^{(10)} + Z_{r4}^{(10)}, \tag{5.46}
$$

with

$$Z_{r1}^{(10)} = \sum_{n=1}^{N} \delta_n^r \frac{\partial P_n}{\partial \dot{b}_{1n}} (\alpha_2 \frac{\partial f_1^2}{\partial P_n} + \alpha_4 \frac{\partial f_1^4}{\partial P_n} + \alpha_5 \frac{\partial f_1^5}{\partial P_n} + \alpha_8 \frac{\partial f_1^8}{\partial P_n}$$
$$+\alpha_9 \frac{\partial f_1^9}{\partial P_n} + \alpha_{10} \frac{\partial f_1^{10}}{\partial P_n})$$
$$Z_{r2}^{(10)} = \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial Q_n}{\partial \dot{c}_{1n}} (\alpha_2 \frac{\partial f_1^2}{\partial Q_n} + \alpha_4 \frac{\partial f_1^4}{\partial Q_n} + \alpha_5 \frac{\partial f_1^5}{\partial Q_n} + \alpha_8 \frac{\partial f_1^8}{\partial Q_n}$$
$$+\alpha_9 \frac{\partial f_1^9}{\partial Q_n} + \alpha_{10} \frac{\partial f_1^{10}}{\partial Q_n}) \tag{5.47}$$
$$Z_{r3}^{(10)} = \sum_{n=1}^{N} \delta_{n+2N+1}^r \frac{\partial B_n}{\partial \dot{b}_{2n}} (\alpha_1 \frac{\partial f_1^1}{\partial B_n} + \alpha_2 \frac{\partial f_1^2}{\partial B_n} + \alpha_3 \frac{\partial f_1^3}{\partial B_n} + \alpha_4 \frac{\partial f_1^4}{\partial B_n}$$
$$+\alpha_5 \frac{\partial f_1^5}{\partial B_n} + \alpha_6 \frac{\partial f_1^6}{\partial B_n})$$
$$Z_{r4}^{(10)} = \sum_{n=1}^{N} \delta_{n+3N+1}^r \frac{\partial C_n}{\partial \dot{c}_{2n}} (\alpha_1 \frac{\partial f_1^1}{\partial C_n} + \alpha_2 \frac{\partial f_1^2}{\partial C_n} + \alpha_3 \frac{\partial f_1^3}{\partial C_n} + \alpha_4 \frac{\partial f_1^4}{\partial C_n}$$
$$+\alpha_5 \frac{\partial f_1^5}{\partial C_n} + \alpha_6 \frac{\partial f_1^6}{\partial C_n})$$

The derivative of the constant term for the torsional motion is

$$Z_r^{(20)} = Z_{r1}^{(20)} + Z_{r2}^{(20)} + Z_{r3}^{(20)} + Z_{r4}^{(20)}, \tag{5.48}$$

With

$$Z_{r1}^{(20)} = \sum_{n=1}^{N} \delta_n^r \frac{\partial P_n}{\partial \dot{b}_{1n}} (\beta_2 \frac{\partial f_1^2}{\partial P_n} + \beta_4 \frac{\partial f_1^4}{\partial P_n} + \beta_5 \frac{\partial f_1^5}{\partial P_n} + \beta_8 \frac{\partial f_1^8}{\partial P_n}$$
$$+\beta_9 \frac{\partial f_1^9}{\partial P_n} + \beta_{10} \frac{\partial f_1^{10}}{\partial P_n})$$
$$Z_{r2}^{(20)} = \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial Q_n}{\partial \dot{c}_{1n}} (\beta_2 \frac{\partial f_1^2}{\partial Q_n} + \beta_4 \frac{\partial f_1^4}{\partial Q_i} + \beta_5 \frac{\partial f_1^5}{\partial Q_i} + \beta_8 \frac{\partial f_1^8}{\partial Q_i}$$
$$+\beta_9 \frac{\partial f_1^9}{\partial Q_i} + \beta_{10} \frac{\partial f_1^{10}}{\partial Q_i})$$

$$Z_{r3}^{(20)} = \sum_{n=1}^{N} \delta_{n+2N+1}^{r} \frac{\partial B_n}{\partial \dot{b}_{2n}} (\beta_1 \frac{\partial f_1^1}{\partial B_n} + \beta_2 \frac{\partial f_1^2}{\partial B_n} + \beta_3 \frac{\partial f_1^3}{\partial B_n} + \beta_4 \frac{\partial f_1^4}{\partial B_n}$$
$$+ \beta_5 \frac{\partial f_1^5}{\partial B_n} + \beta_6 \frac{\partial f_1^6}{\partial B_n}) \tag{5.49}$$
$$Z_{r4}^{(20)} = \sum_{n=1}^{N} \delta_{n+3N+1}^{r} \frac{\partial C_n}{\partial \dot{c}_{2n}} (\beta_1 \frac{\partial f_1^1}{\partial C_n} + \beta_2 \frac{\partial f_1^2}{\partial C_n} + \beta_3 \frac{\partial f_1^3}{\partial C_n} + \beta_4 \frac{\partial f_1^4}{\partial C_n}$$
$$+ \beta_5 \frac{\partial f_1^5}{\partial C_n} + \beta_6 \frac{\partial f_1^6}{\partial C_n})$$

The derivative of the cosine term for the transverse motion is

$$Z_{kr}^{(1c)} = Z_{kr1}^{(1c)} + Z_{kr2}^{(1c)} + Z_{kr3}^{(1c)} + Z_{kr4}^{(1c)} \tag{5.50}$$

with

$$Z_{kr1}^{(1c)} = \sum_{n=1}^{N} \delta_n^r (\frac{\partial P_n}{\partial \dot{b}_{1n}} (\alpha_2 \frac{\partial f_2^2}{\partial P_n} + \alpha_4 \frac{\partial f_2^4}{\partial P_n} + \alpha_5 \frac{\partial f_2^5}{\partial P_n} + \alpha_8 \frac{\partial f_2^8}{\partial P_n} + \alpha_9 \frac{\partial f_2^9}{\partial P_n}$$
$$+ \alpha_{10} \frac{\partial f_2^{10}}{\partial P_n}) - \delta_n^k [\alpha_{11} + \alpha_{14}])$$
$$Z_{kr2}^{(1c)} = \sum_{n=1}^{N} \delta_{n+N}^{r} (\frac{\partial Q_n}{\partial \dot{c}_{1n}} (\alpha_2 \frac{\partial f_2^2}{\partial Q_n} + \alpha_4 \frac{\partial f_2^4}{\partial Q_n} + \alpha_5 \frac{\partial f_2^5}{\partial Q_n} + \alpha_8 \frac{\partial f_2^8}{\partial Q_n} + \alpha_9 \frac{\partial f_2^9}{\partial Q_n}$$
$$+ \alpha_{10} \frac{\partial f_2^{10}}{\partial Q_n}) - \delta_n^k 2k\Omega) \tag{5.51}$$
$$Z_{kr3}^{(1c)} = \sum_{n=1}^{N} \delta_{n+2N+1}^{r} (\frac{\partial B_n}{\partial \dot{b}_{2n}} (\alpha_1 \frac{\partial f_2^1}{\partial B_n} + \alpha_2 \frac{\partial f_2^2}{\partial B_n} + \alpha_3 \frac{\partial f_2^3}{\partial B_n} + \alpha_4 \frac{\partial f_2^4}{\partial B_n} + \alpha_5 \frac{\partial f_2^5}{\partial B_n}$$
$$+ \alpha_6 \frac{\partial f_2^6}{\partial B_n}) - \delta_n^k \alpha_7)$$
$$Z_{kr4}^{(1c)} = \sum_{n=1}^{N} \delta_{n+3N+1}^{r} \frac{\partial C_n}{\partial \dot{c}_{2n}} (\alpha_1 \frac{\partial f_2^1}{\partial C_n} + \alpha_2 \frac{\partial f_2^2}{\partial C_n} + \alpha_3 \frac{\partial f_2^3}{\partial C_n} + \alpha_4 \frac{\partial f_2^4}{\partial C_n} + \alpha_5 \frac{\partial f_2^5}{\partial C_n}$$
$$+ \alpha_6 \frac{\partial f_2^6}{\partial C_n})$$

The derivative of the cosine term for the torsional motion is

$$Z_{kr}^{(2c)} = Z_{kr1}^{(2c)} + Z_{kr2}^{(2c)} + Z_{kr3}^{(2c)} + Z_{kr4}^{(2c)} \tag{5.52}$$

with

$$Z_{kr1}^{(2c)} = \sum_{n=1}^{N} \delta_n^r (\frac{\partial P_i}{\partial \dot{b}_{1n}} (\beta_2 \frac{\partial f_2^2}{\partial P_n} + \beta_4 \frac{\partial f_2^4}{\partial P_n} + \beta_5 \frac{\partial f_2^5}{\partial P_n} + \beta_8 \frac{\partial f_2^8}{\partial P_n} + \beta_9 \frac{\partial f_2^9}{\partial P_n}$$

$$+ \beta_{10} \frac{\partial f_2^{10}}{\partial P_n}) - \delta_n^k \beta_{11})$$

$$Z_{kr2}^{(2c)} = \sum_{n=1}^{N} \delta_{n+N}^r \frac{\partial Q_n}{\partial \dot{c}_{1n}} (\beta_2 \frac{\partial f_2^2}{\partial Q_n} + \beta_4 \frac{\partial f_2^4}{\partial Q_n} + \beta_5 \frac{\partial f_2^5}{\partial Q_n} + \beta_8 \frac{\partial f_2^8}{\partial Q_n} + \beta_9 \frac{\partial f_2^9}{\partial Q_n}$$

$$+ \beta_{10} \frac{\partial f_2^{10}}{\partial Q_n}) \tag{5.53}$$

$$Z_{kr3}^{(2c)} = \sum_{n=1}^{N} \delta_{n+2N+1}^r (\frac{\partial B_n}{\partial \dot{b}_{2n}} (\beta_1 \frac{\partial f_2^1}{\partial B_n} + \beta_2 \frac{\partial f_2^2}{\partial B_n} + \beta_3 \frac{\partial f_2^3}{\partial B_n} + \beta_4 \frac{\partial f_2^4}{\partial B_n} + \beta_5 \frac{\partial f_2^5}{\partial B_n}$$

$$+ \beta_6 \frac{\partial f_2^6}{\partial B_n}) - \delta_n^k [\beta_7 + \beta_{14}])$$

$$Z_{kr4}^{(2c)} = \sum_{n=1}^{N} \delta_{n+3N+1}^r (\frac{\partial C_n}{\partial \dot{c}_{2n}} (\beta_1 \frac{\partial f_2^1}{\partial C_n} + \beta_2 \frac{\partial f_2^2}{\partial C_n} + \beta_3 \frac{\partial f_2^3}{\partial C_n} + \beta_4 \frac{\partial f_2^4}{\partial C_n} +$$

$$\beta_5 \frac{\partial f_2^5}{\partial C_n} + \beta_6 \frac{\partial f_2^6}{\partial C_n}) - \delta_n^k [2k\Omega])$$

The derivative of the sine term for the transverse motion is

$$Z_{kr}^{(1s)} = Z_{kr1}^{(1s)} + Z_{kr2}^{(1s)} + Z_{kr3}^{(1s)} + Z_{kr4}^{(1s)} \tag{5.54}$$

with

$$Z_{kr1}^{(1s)} = \sum_{n=1}^{N} \delta_n^r (\frac{\partial P_n}{\partial \dot{b}_{1n}} (\alpha_2 \frac{\partial f_3^2}{\partial P_n} + \alpha_4 \frac{\partial f_3^4}{\partial P_n} + \alpha_5 \frac{\partial f_3^5}{\partial P_n} + \alpha_8 \frac{\partial f_3^8}{\partial P_n} + \alpha_9 \frac{\partial f_3^9}{\partial P_n} +$$

$$\alpha_{10} \frac{\partial f_3^{10}}{\partial P_n}) + \delta_n^k 2k\Omega)$$

$$Z_{kr2}^{(1s)} = \sum_{n=1}^{N} \delta_{n+N}^r (\frac{\partial Q_n}{\partial \dot{c}_{1n}} (\alpha_2 \frac{\partial f_3^2}{\partial Q_n} + \alpha_4 \frac{\partial f_3^4}{\partial Q_n} + \alpha_5 \frac{\partial f_3^5}{\partial Q_n} + \alpha_8 \frac{\partial f_3^8}{\partial Q_n} + \alpha_9 \frac{\partial f_3^9}{\partial Q_n}$$

$$+\alpha_{10} \frac{\partial f_3^{10}}{\partial Q_n}) - \delta_n^k [\alpha_{11} + \alpha_{14}])$$

$$Z_{kr3}^{(1s)} = \sum_{n=1}^{N} \delta_{n+2N+1}^{r} \frac{\partial B_n}{\partial \dot{b}_{2n}} (\alpha_1 \frac{\partial f_3^1}{\partial B_n} + \alpha_2 \frac{\partial f_3^2}{\partial B_n} + \alpha_3 \frac{\partial f_3^3}{\partial B_n} + \alpha_4 \frac{\partial f_3^4}{\partial B_n} + \alpha_5 \frac{\partial f_3^5}{\partial B_n}$$
$$+\alpha_6 \frac{\partial f_3^6}{\partial B_n}) \tag{5.55}$$
$$Z_{kr4}^{(1s)} = \sum_{n=1}^{N} \delta_{n+3N+1}^{r} (\frac{\partial C_n}{\partial \dot{c}_{2n}} (\alpha_1 \frac{\partial f_3^1}{\partial C_n} + \alpha_2 \frac{\partial f_3^2}{\partial C_n} + \alpha_3 \frac{\partial f_3^3}{\partial C_n} + \alpha_4 \frac{\partial f_3^4}{\partial C_n} + \alpha_5 \frac{\partial f_3^5}{\partial C_n}$$
$$+\alpha_6 \frac{\partial f_3^6}{\partial C_n}) - \delta_n^k \alpha_7)$$

The derivative of the sine term for the torsional motion is

$$Z_{kr}^{(2s)} = Z_{kr1}^{(2s)} + Z_{kr2}^{(2s)} + Z_{kr3}^{(2s)} + Z_{kr4}^{(2s)} \tag{5.56}$$

with

$$Z_{kr1}^{(2s)} = \sum_{n=1}^{N} \delta_n^r \frac{\partial P_n}{\partial \dot{b}_{1n}} (\beta_2 \frac{\partial f_3^2}{\partial P_n} + \beta_4 \frac{\partial f_3^4}{\partial P_n} + \beta_5 \frac{\partial f_3^5}{\partial P_n} + \beta_8 \frac{\partial f_3^8}{\partial P_n} + \beta_9 \frac{\partial f_3^9}{\partial P_n} +$$
$$\beta_{10} \frac{\partial f_3^{10}}{\partial P_n})$$
$$Z_{kr2}^{(2s)} = \sum_{n=1}^{N} \delta_{n+N}^{r} (\frac{\partial Q_n}{\partial \dot{c}_{1n}} (\beta_2 \frac{\partial f_3^2}{\partial Q_n} + \beta_4 \frac{\partial f_3^4}{\partial Q_n} + \beta_5 \frac{\partial f_3^5}{\partial Q_n} + \beta_8 \frac{\partial f_3^8}{\partial Q_n} + \beta_9 \frac{\partial f_3^9}{\partial Q_n}$$
$$+\beta_{10} \frac{\partial f_3^{10}}{\partial Q_n}) - \delta_n^k \beta_{11}) \tag{5.57}$$
$$Z_{kr3}^{(2s)} = \sum_{n=1}^{N} \delta_{n+2N+1}^{r} (\frac{\partial B_n}{\partial \dot{b}_{2n}} (\beta_1 \frac{\partial f_3^1}{\partial B_n} + \beta_2 \frac{\partial f_3^2}{\partial B_n} + \beta_3 \frac{\partial f_3^3}{\partial B_n} + \beta_4 \frac{\partial f_3^4}{\partial B_n} + \beta_5 \frac{\partial f_3^5}{\partial B_n}$$
$$+\beta_6 \frac{\partial f_3^6}{\partial B_n}) + \delta_n^k 2k\Omega)$$
$$Z_{kr4}^{(2s)} = \sum_{n=1}^{N} (\delta_{n+3N+1}^{r} \frac{\partial C_n}{\partial \dot{c}_{2n}} (\beta_1 \frac{\partial f_3^1}{\partial C_n} + \beta_2 \frac{\partial f_3^2}{\partial C_n} + \beta_3 \frac{\partial f_3^3}{\partial C_n} + \beta_4 \frac{\partial f_3^4}{\partial C_n} + \beta_5 \frac{\partial f_3^5}{\partial C_n} +$$
$$\beta_6 \frac{\partial f_3^6}{\partial C_n}) - \delta_n^k [\beta_7 + \beta_{14}])$$

From Luo [2012], the eigenvalues of $D\mathbf{f}^{(m)}(\mathbf{y}^{*(m)})$ are classified as

$$(n_1, n_2, n_3 \,|\, n_4, n_5, n_6) \tag{5.58}$$

The corresponding boundary between the stable and unstable solutions is given by the saddle-node bifurcation and Hopf bifurcation.

5.2 Analytical routes to chaos

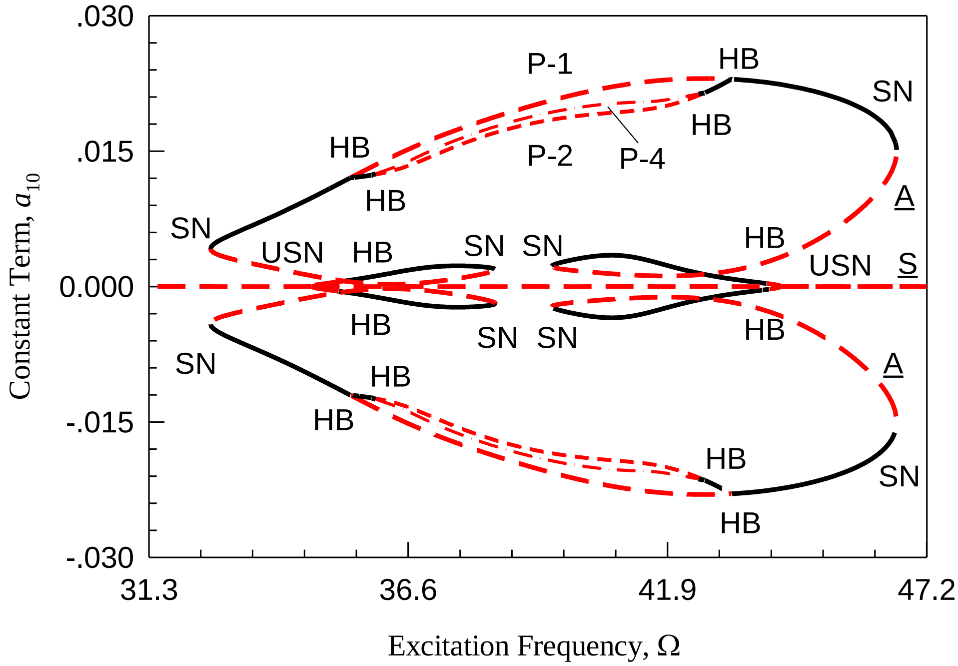
Consider the following parameters:

$$\begin{aligned} &\zeta_y = 0.0037,\ \zeta_\theta = 0.0046,\ \eta_y = 0.000922,\ \eta_\theta = 0.0062 \\ &a_1 = 2.341,\ a_3 = 14.366,\ b_1 = 0.496,\ b_3 = 1.265 \\ &U = 6.77,\ \rho = 1.255,\ Q_0 = 100.0,\ k_y^{'} = 5.0,\ k_\theta^{'} = 2.0, \end{aligned} \tag{5.60}$$

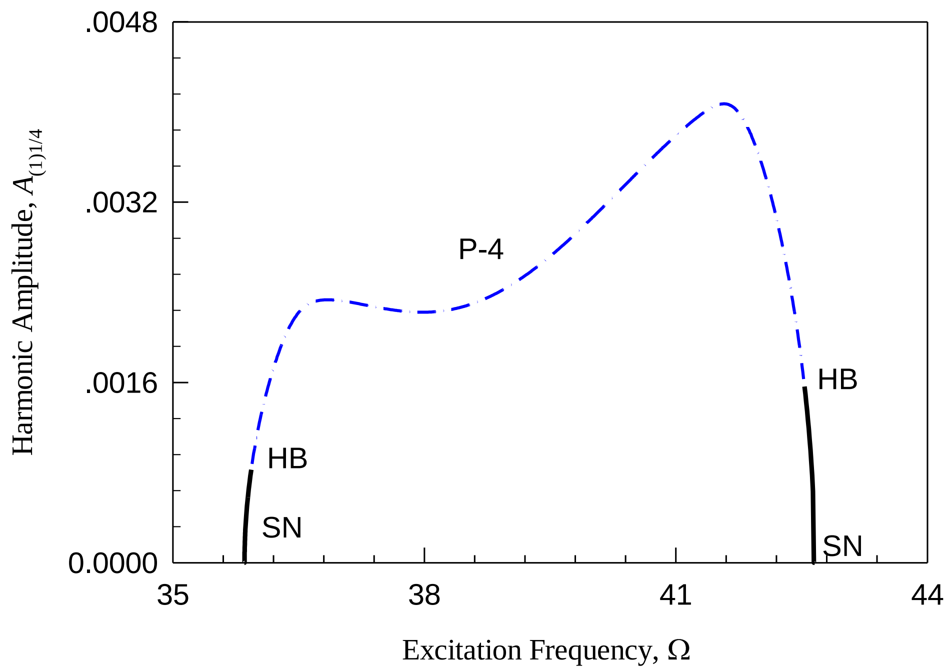
Where

$$w_y = \sqrt{k_y / \mathfrak{M}},\ w_\theta = \sqrt{k_\theta / I},\ \zeta_y = \frac{c_y}{2\mathfrak{M} w_y},\ \zeta_\theta = \frac{c_\theta}{2 I w_\theta},\ \eta_y = \frac{\rho d^2}{2\mathfrak{M}},\ \eta_\theta = \frac{\rho d^4}{2I} \tag{5.61}$$

The acronym "SN" and "USN" represent the stable and unstable saddle-node bifurcations. The acronym "HB" represents the Hopf bifurcation (supercritical). "A" and "S" denote asymmetric and symmetric period-1 motions respectively. "P-m" indicates one period of the motion is "m" times the time of one excitation period. Solid and dashed curves represent stable and unstable period-m motions, respectively. From the above the parameters, the frequency-amplitude curves of period-1 to period-4 motion in transverse-direction and torsional-direction of such a nonlinear cable are presented in Figure 5.1 and 5.2 that are based on 120 harmonic terms.



(i)

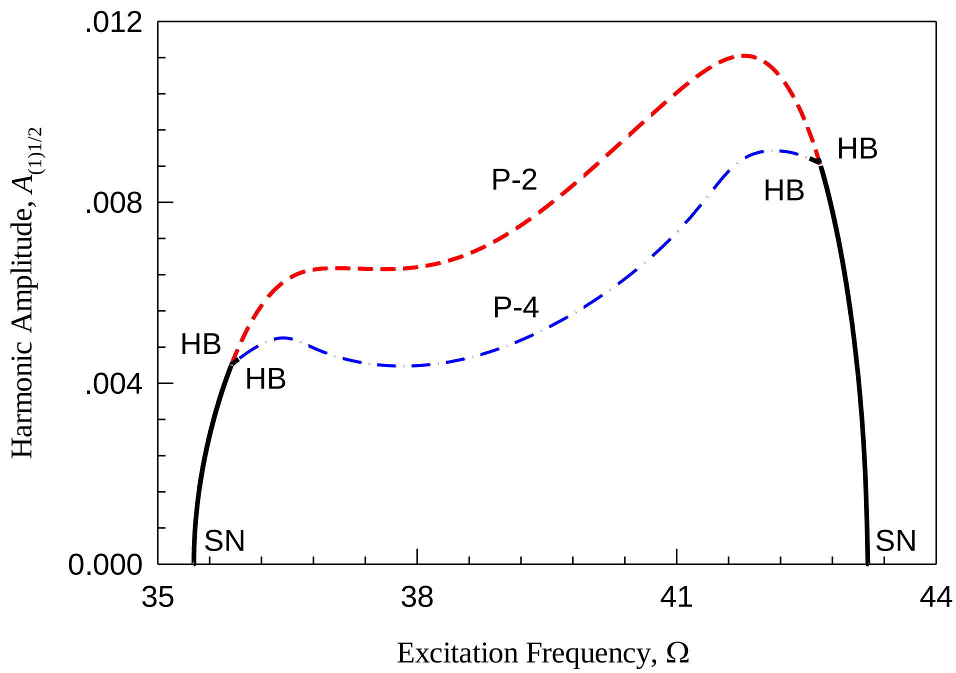


(ii)

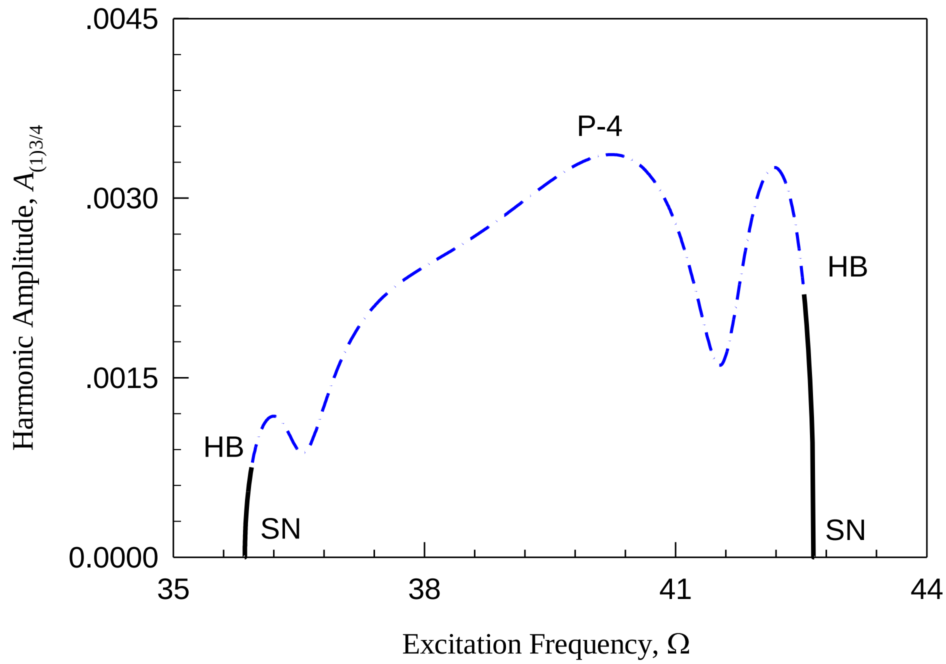
Figure 5.1: Bifurcation tree from period-1 motion to chaos of the nonlinear cable structure in the transverse direction: frequency-amplitude curves of harmonic terms based on 30 harmonic terms (HB30): (i) $a_{10}^{(m)}$, (ii)-(xvi) $A_{(1)k/m}$ ($m = 4, k = 1, 2, ..., 4; 8, 12, 16, 20, 24; 116, 117, 118, 119, 120$),

($\zeta_y = 0.0037, \zeta_\theta = 0.0046, \eta_y = 0.000922, \eta_\theta = 0.0062, a_1 = 2.341, a_3 = 14.366, b_1 = 0.496,$

$b_3 = 1.265, U = 6.77, \rho = 1.255, d = 33\times 10^{-3}, k_y^{'} = 5.0, k_\theta^{'} = 2.0, Q_0 = 100$)



(iii)



(iv)

Figure 5.1 Continued

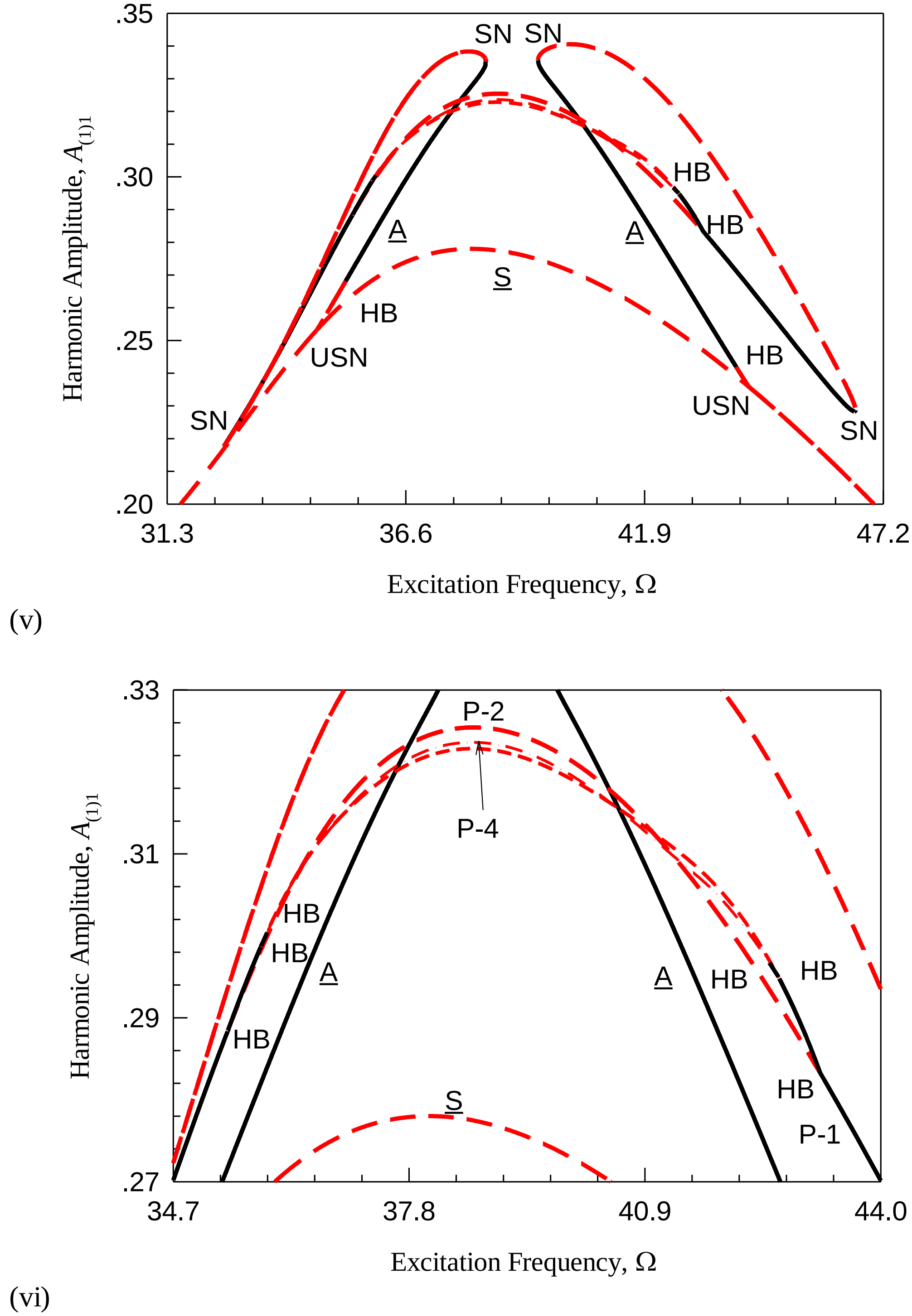
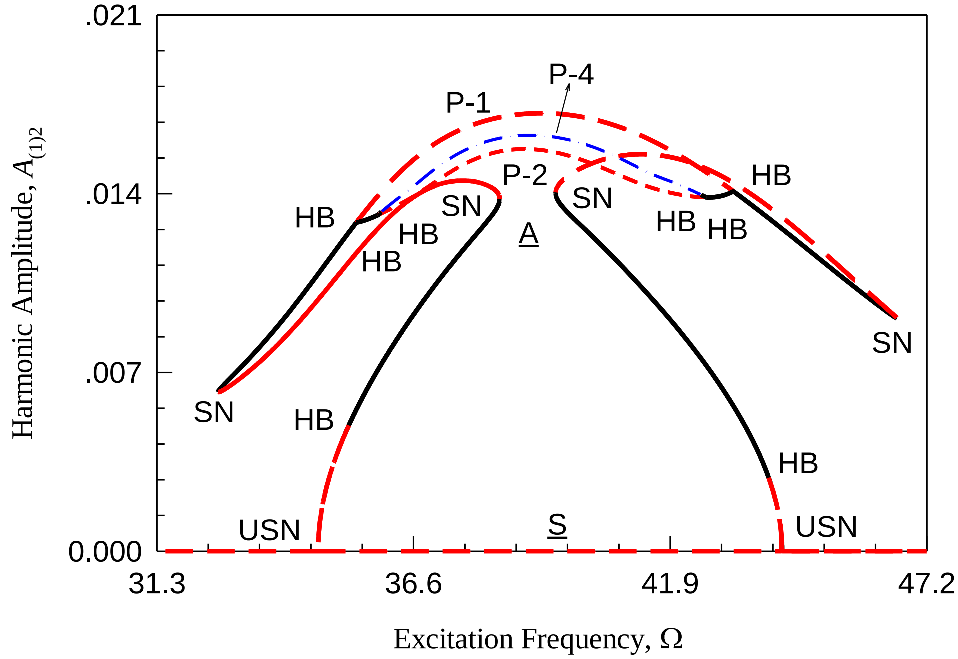
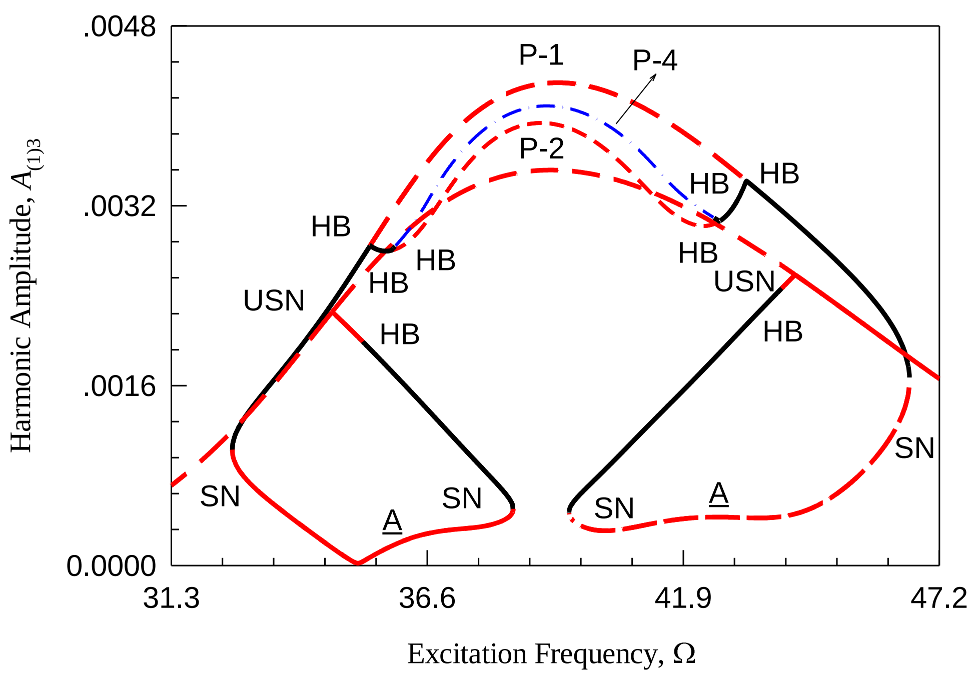


Figure 5.1 Continued

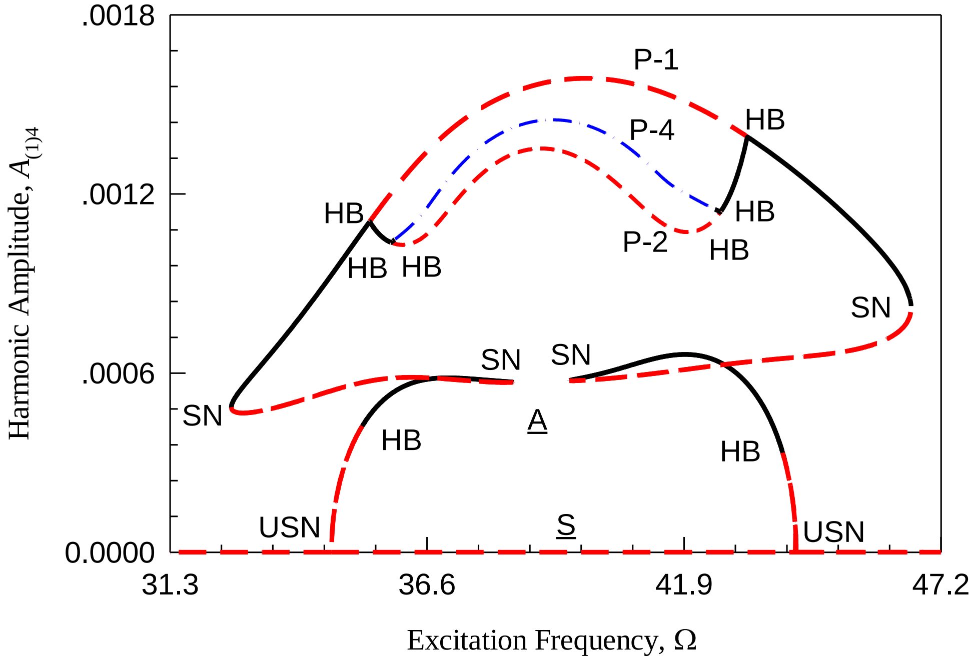


(vii)

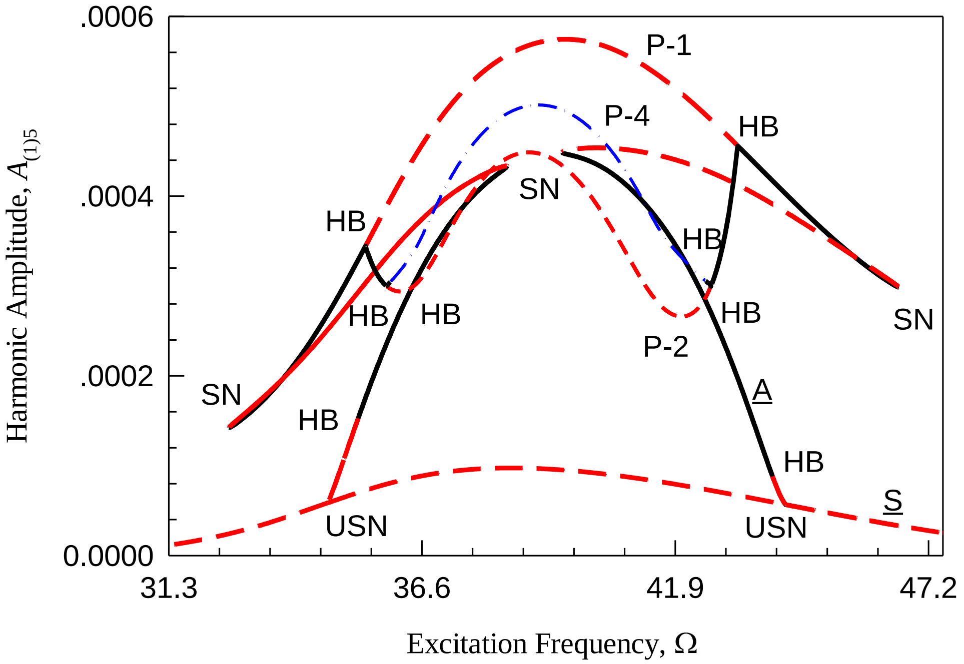


(viii)

Figure 5.1 Continued

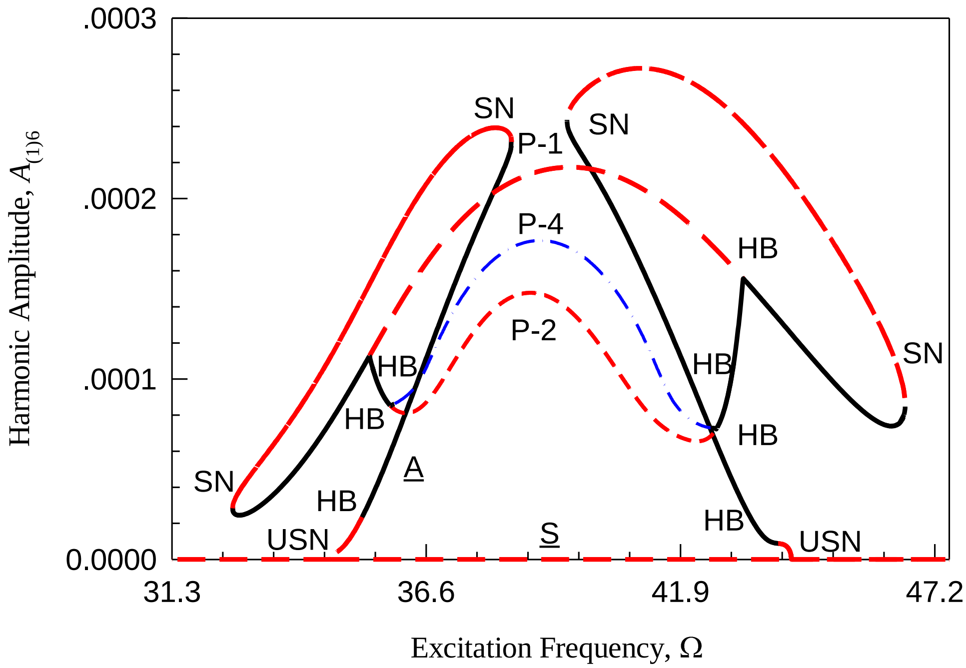


(ix)

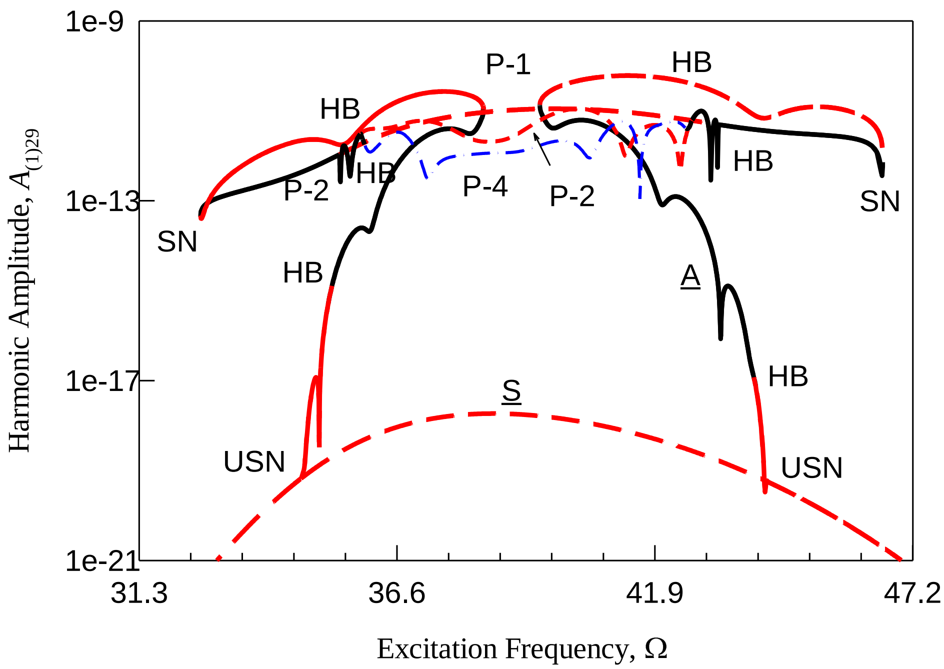


(x)

Figure 5.1 Continued

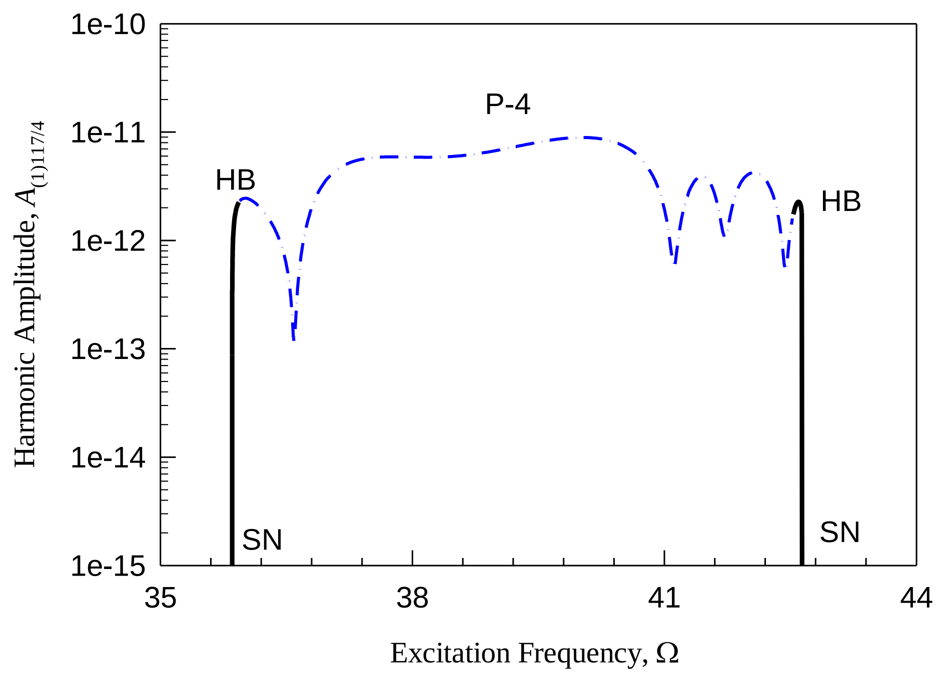


(xi)

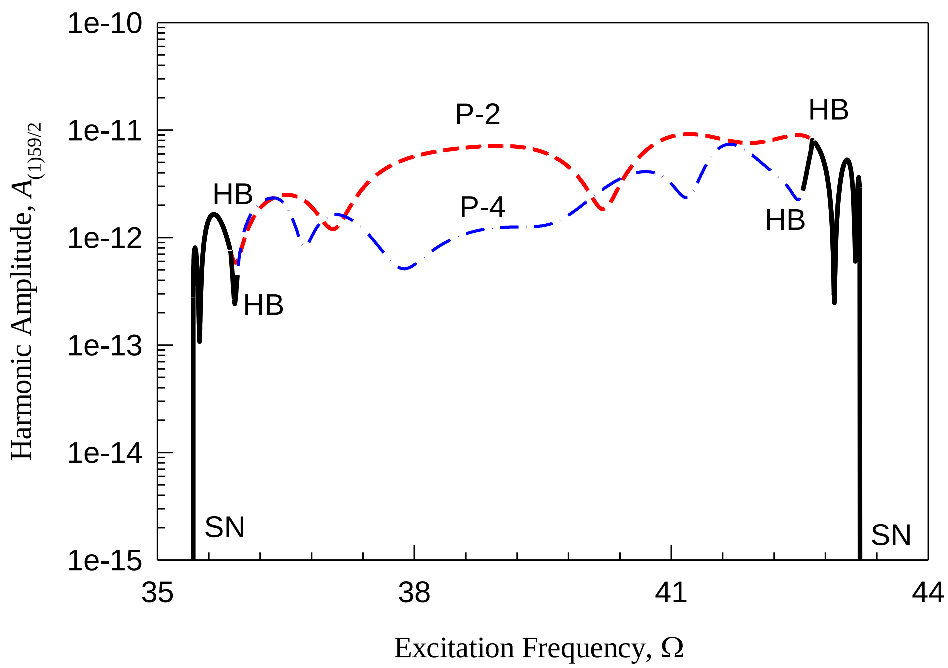


(xii)

Figure 5.1 Continued

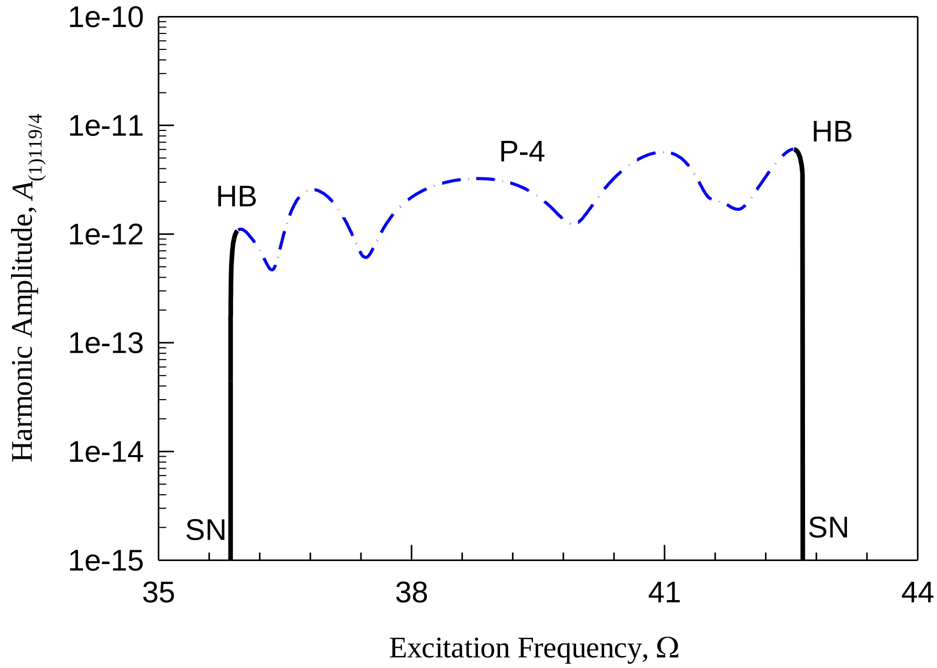


(xiii)

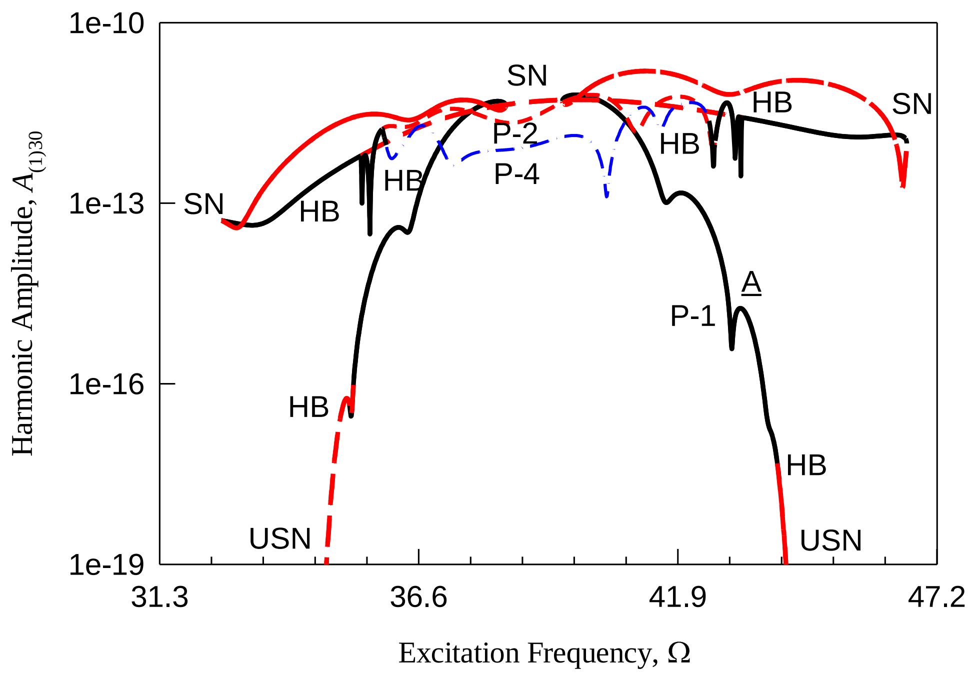


(xiv)

Figure 5.1 Continued



(xv)



(xvi)

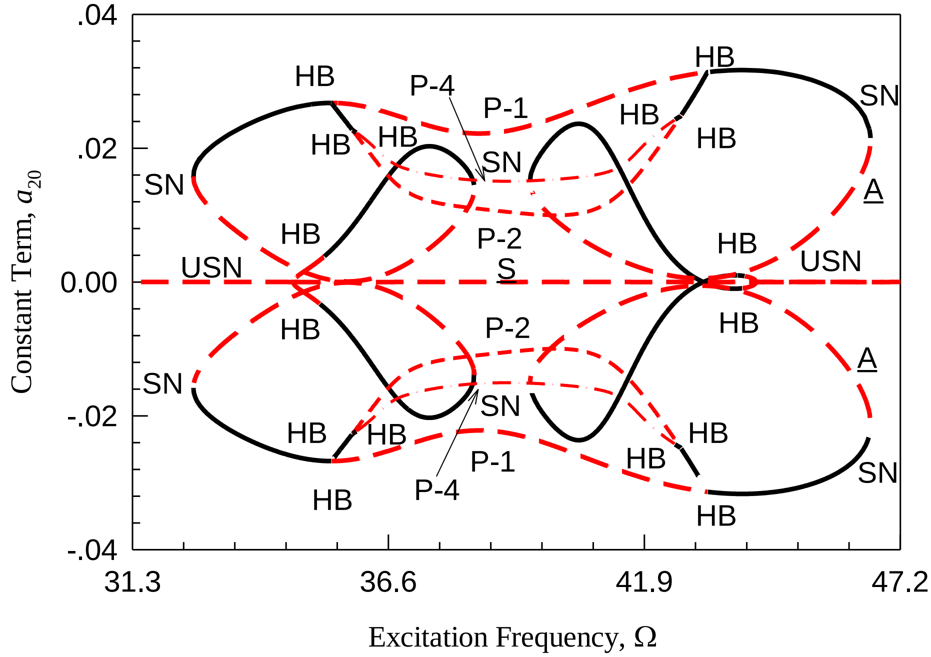
Figure 5.1 Continued

In Figures 5.1, the bifurcation trees of the period-1 to period-4 motion in the transverse direction is presented through the frequency-amplitude curves. In Figure 5.1 (i), the constant $a_{10}^{(m)}$ versus excitation frequency Ω is presented. Both symmetric and asymmetric period-1 motions can be found for $\Omega \in (31.3, 47.2)$. The symmetric period-1 motions are all unstable and the $a_{10}^{(m)} = 0$. The unstable saddle node bifurcations (USN) of symmetric period-1 motions are at $\Omega \approx 44.2135, 34.6338$ where unstable asymmetric period-1 motions can be observed. The Hopf bifurcations (HB) of stable asymmetric periods motions are at $\Omega \approx 44.1, 35.17$ and $\Omega \approx 43.21,.$ 35.42. The asymmetrical period-1 motions become quasi-periodic or chaotic motions at the first two Hopf bifurcation(HB) points. At the other two bifurcation points, period of the period-1 motion doubles. The saddle node bifurcations (SN) of asymmetric period-1 motions are at $\Omega \approx 32.56, 38.37, 39.53, 46.56$ where stable asymmetric period-1 motions disappear. Meanwhile period-2 and period-4 motions can also be seen in this figure. The stable period-2 and period-4 motions are represented in black solid lines that can be observed at the Hopf (HB) bifurcations of period-1 and period-2 motions respectively. The unstable period-2 and period-4 motions are depicted in short dash lines and dash dot lines correspondingly. The constant terms are symmetrical about the $a_{10}^{(m)} = 0$. In Figure 5.1 (ii), the harmonic amplitude $A_{(1)1/4}$ varying with excitation frequency Ω is presented. Only one branch of period-4 motions are observed for this frequency range. For the period-1 and period-2 motions, $A_{(1)1/4} = 0.$ The saddle-node (SN) bifurcations of the period-4 motions are at $\Omega \approx 35.8548, 42.6450$ where stable period-4 motions end. The Hopf bifurcations (HB) of the period-4 motion are at $\Omega \approx 35.935, 42.535$ from which period-8 motions can be obtained. The unstable period-4 motions are presented by the dot-dashed lines. The quantity level of such a harmonic amplitude is $A_{(1)1/4} \sim 4.8 \times 10^{-3}.$ In

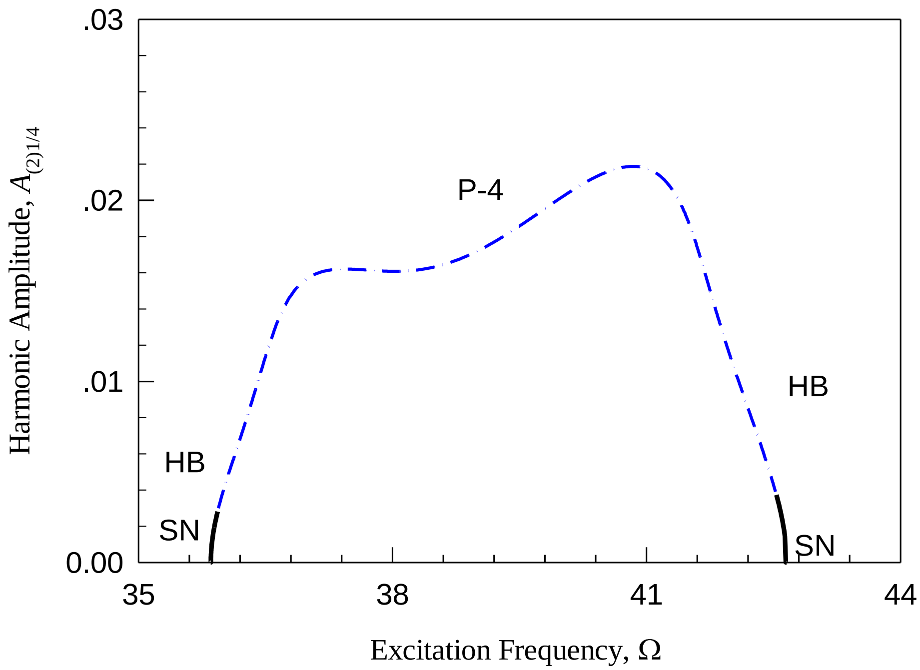
Figure 5.1 (iii), the harmonic amplitude $A_{(1)1/2}$ versus excitation frequency Ω is presented. The period-2 and period-4 motions on the bifurcation tree can be observed. For period-1 motions, $A_{(1)1/2} = 0.$ One branch of period-2 motions is observed. The saddle-node (SN) bifurcations of period-2 motions are at $\Omega \approx 35.417, 43.22$. The Hopf bifurcations of period-2 motions are at $\Omega \approx 35.8548, 42.6450$ which are also the saddle-node (SN) bifurcations of period-4 motions in Fig.2(ii). The quantity level of the harmonic amplitude is $A_{(1)1/2} \sim 0.012.$ In Figure 5.1 (iv), harmonic amplitude $A_{(1)3/4}$ versus excitation frequency is presented, which is similar to $A_{(1)1/4}$. The quantity levels for both $A_{(1)1/4}$ and $A_{(1)3/4}$ are quite close, i.e., $A_{(1)1/4} \sim 4.8\times10^{-3}$ and $A_{(1)3/4} \sim 4.5\times10^{-3}.$ However, the variation of harmonic amplitudes with excitation frequency are different for the two harmonic amplitudes $A_{(1)1/4}$ and $A_{(1)3/4}$. In Figure 5.1 (v), the primary harmonic amplitude $A_{(1)1}$ varying with excitation frequency is presented for period-1 to period-4 motions. One zoomed Figure 5.1 (vi) for $\Omega \in (34.7, 44.0)$ is presented to better present the bifurcation scene of periodic motions. For both symmetric and asymmetric period-1 motions, the value of harmonic term is not equal to zero. The saddle-node bifurcations of symmetric period-1 motions are both unstable at $\Omega \approx 44.21 \quad 34.63$. They are the onset points of asymmetric unstable period-1 motions, which are also for the unstable saddle-node bifurcations for asymmetric period-1 motions. The Hopf bifurcations of asymmetric period-1 motions are at $\Omega \approx 44.1, 43.21, 35.42, 35.17$. The quantity level of the primary harmonic amplitude is $A_{(1)1} \sim 0.35.$ The symmetric period-1 motions exist for other frequency range. To avoid abundant illustrations, only a few main harmonic amplitudes are presented herein. Thus, in Figure 5.1 (vii), the harmonic amplitude of $A_{(1)2}$ varying with excitation frequency is presented for $\Omega \in (31.3, 47.2)$. For symmetric period-1 motions, $A_{(1)2} = 0$. For asymmetric period-1 motions, $A_{(1)2} \neq 0$, and the corresponding bifurcation trees can be observed. For higher frequency area, periodic motions with different periods may also be found. The quantity level of the second harmonic amplitudes is about $A_{(1)2} \sim 0.021$. In Figure 5.1 (viii), the harmonic amplitude $A_{(1)3}$ versus excitation frequency is presented, which is different from the primary harmonic amplitude $A_{(1)1}$. The bifurcation trees from period-1 motions to period-4 motions can be found on the top area. The lower half of the figure shows the branch of symmetric (S) and asymmetric (A) period-1 motions. The asymmetrical period-1 motions switch from stable to unstable at the Hopf bifurcations (HB) points. The quantity level of the third harmonic amplitudes is about $A_{(1)3} \sim 0.0048$. To compare with the harmonic amplitude $A_{(1)2}$, the harmonic amplitude $A_{(1)4}$ varying with excitation frequency is presented in Figure 5.1 (ix). The symmetric period-1 motions possess $A_{(1)4} = 0$. The asymmetric period-1 motion of $A_{(1)4} \neq 0$ experiences the bifurcation trees in such frequency-amplitude curves. The quantity level of the fourth harmonic amplitudes is $A_{(1)4} \sim 0.0018$. As harmonic order increases, the harmonic amplitude decays for the same frequency range. Thus the harmonic amplitude $A_{(1)5}$ versus excitation frequency is presented in Figure 5.1 (x). The pattern of the bifurcation tree is very different from $A_{(1)3}$ and $A_{(1)1}$. The quantity level of the harmonic amplitude is $A_{(1)5} \sim 6\times10^{-3}$ for $\Omega \in (31.3, \quad 47.2)$. The harmonic amplitude $A_{(1)6}$ versus excitation frequency is presented in Figure 5.1 (xi) to be compared with other even terms. The shape of the bifurcation tree is also quite different from $A_{(1)2}$ and $A_{(1)4}$. The quantity level of the harmonic amplitude is $A_{(1)6} \sim 3\times10^{-3}$. To demonstrate the accuracy of the analytical solutions, the last set of harmonic amplitudes are discussed. In Figure 5.1 (xii), the harmonic amplitude $A_{(1)29}$ varying with excitation frequency is presented. The ordinate is in common logarithmic scale instead of linear to exhibit the drastic drop of the magnitude. The quantity level of the harmonic amplitude is from $A_{(1)29} \sim 10^{-9}$ as excitation frequency varies from $\Omega = 31.3$ to $\Omega = 47.2$ for period-1 to period-4 motions. The harmonic amplitude $A_{(1)117/4}$ versus excitation amplitude is presented in Figure 5.1 (xiii) for period-4 motions. From one branch, the quantity level of harmonic amplitude is $A_{(1)117/4} \sim 10^{-10}$ for excitation frequency varies from $\Omega = 35$ to $\Omega = 44$. In Figure 5.1 (xiv), the harmonic amplitude $A_{(1)59/2}$ versus excitation amplitude is presented for period-2 and period-4 motions on the bifurcation tree. The quantity level of harmonic amplitude is under $A_{(1)59/2} \sim 10^{-10}$. Similarly, in Figure 5.1 (xv), the harmonic amplitude $A_{(1)119/4}$ versus excitation amplitude is presented for period-4 motions, and the quantity level of harmonic amplitude is also under $A_{(1)117/4} \sim 10^{-10}$. In Figure 5.1 (xvi), the harmonic amplitude varying with excitation frequency is presented for period-1 to period-4 motions. The quantity level of harmonic amplitude $A_{(1)30} \sim 10^{-10}$. Compared to the unstable solutions, the analytical solutions of periodic stable motions are much accurate in such excitation frequency.

For the analytical solutions of periodic motion in the torsional direction, they are also discussed. The bifurcation locations of torsional motions are the same as the transverse direction. In Figure 5.2, the bifurcation trees of the period-1 to period-4 motion in torsional direction is presented through the frequency-amplitude curves. In Figure 5.2 (i), the constant $a_{20}^{(m)}$ versus excitation frequency Ω is presented. For symmetric period-1 motion, $a_{20}^{(m)} = 0$. For asymmetric period-1 to period-4 motion, $a_{20}^{(m)} \neq 0$. In Figure 5.2 (ii), the harmonic amplitude $A_{(2)1/4}$ varying with excitation frequency Ω is presented. For the period-1 and period-2 motions, $A_{(2)1/4} = 0.$ The quantity level of such harmonic amplitude is $A_{(2)1/4} \sim 0.03$. In Figure 5.2 (iii), the harmonic amplitude $A_{(2)1/2}$ varying with excitation frequency Ω is presented. Period-1 motions possess $A_{(1)1/2} = 0.$ One branch of period-2 motions are observed. Period-4 motion exists at the Hopf (HB) bifurcations of period-2 motions. The quantity level of the harmonic amplitudes is $A_{(2)1/2} \sim 0.06$. In Figure 5.2 (iv), the harmonic amplitude $A_{(2)3/4}$ versus excitation frequency is presented. The quality level of harmonic amplitude is $A_{(2)3/4} \sim 0.03$. In Figure 5.2 (v), the primary harmonic amplitude $A_{(2)1}$ varying with excitation frequency is presented for period-1 to period-4 motion. One zoomed Figure 5.2 (vi) is shown to better illustrate the bifurcation tree of period-1 to period-4 motion. The quantity level of the primary harmonic amplitude is $A_{(2)1} \sim 0.6.$ Similar to the transverse direction, only a few primary harmonic amplitudes for the torsional direction are presented. Thus, in Figure 5.2 (vii), the harmonic amplitude of $A_{(2)2}$ varying with excitation frequency is presented for $\Omega \in (31.3, 47.2)$. For symmetric period-1 motions, we have $A_{(2)2} = 0.$ For asymmetric period-1 motions, $A_{(2)2} \neq 0,$ and the corresponding bifurcation trees can be observed. The pattern of the harmonic amplitude $A_{(2)2}$ is also different from $A_{(1)2}$. The quantity level of the second harmonic amplitudes is $A_{(2)2} \sim 0.9$ for $\Omega \in (31.3, 47.2)$. In Figure 5.2 (viii), the harmonic amplitude $A_{(2)3}$ versus excitation frequency is presented, which is not similar to $A_{(1)3}$. The bifurcation trees of period-1 to period-4 motion are observed. To compare with the harmonic amplitude $A_{(2)2}$, the harmonic amplitude $A_{(2)4}$ varying with excitation frequency is presented in Figure 5.2 (ix), which is also not similar to

$A_{(1)4}$ The symmetric period-1 motions possess $A_{(2)4} = 0$. The asymmetric period-1 motion of $A_{(2)4} \neq 0$ experiences the bifurcation tree in such frequency-amplitude curves. The harmonic amplitudes $A_{(2)5}$ and $A_{(2)6}$ versus excitation frequency are presented in Figure 5.2 (x) and (xi) to be compared with $A_{(1)5}$ and $A_{(1)6}$ which the change in magnitude with frequency are quite different. To avoid abundant illustration, the last set of harmonic amplitudes are discussed. In Figure 5.2 (xii), the harmonic amplitude $A_{(2)29}$ varying with excitation frequency is presented, similar to $A_{(1)29}$. The quality level of the harmonic amplitude is $A_{(2)29} \sim 10^{-7}$ as the excitation frequency changes from $\Omega = 31.3$ through $\Omega = 47.2$ for period-1 to period-4 motions. The harmonic amplitude $A_{(2)117/4}$ versus excitation amplitude is presented in Figure 5.2 (xiii) for period-4 motions. The quantity level of harmonic amplitude are $A_{(1)117/4} \sim 10^{-8}$. In Figure 5.2 (xiv), the harmonic amplitude $A_{(2)59/2}$ varying with excitation amplitude is presented for period-2 and period-4 motions on the bifurcation trees. The quantity levels of harmonic amplitude is $A_{(2)59/2} \sim 10^{-8}$. Similarly, in Figure 5.2 (xv), the harmonic amplitude $A_{(2)119/4}$ versus excitation amplitude is presented for period-4 motions, and the quantity level of harmonic amplitude is $A_{(2)117/4} \sim 10^{-8}$. In Figure 5.2 (xvi), the harmonic amplitude $A_{(2)30}$ varying with excitation frequency is presented for period-1 to period-4 motion. The quantity level of harmonic amplitude $A_{(2)30}$ on the three branch is $A_{(2)30} \sim 10^{-8}$.

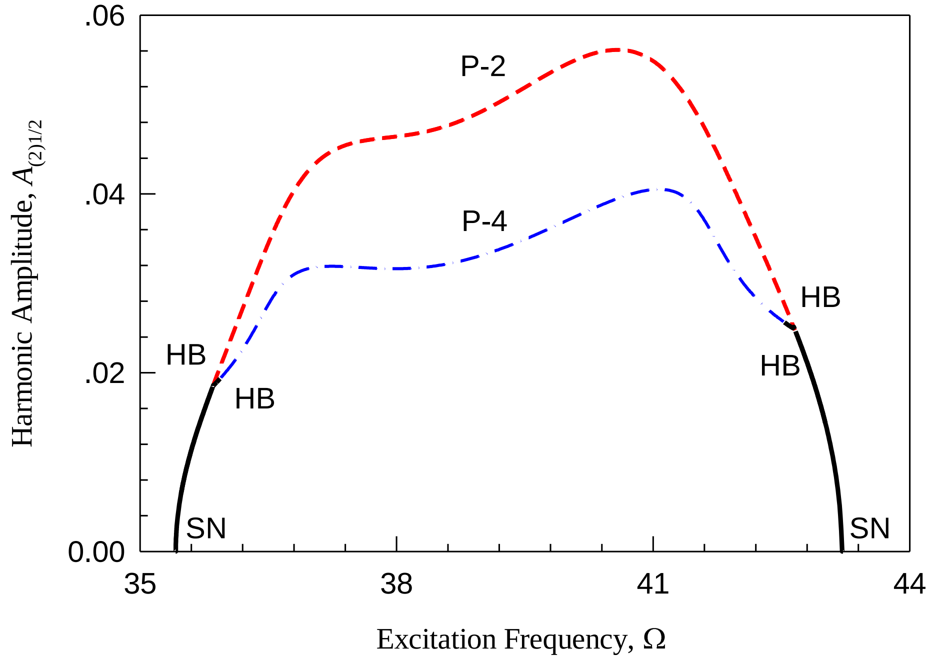


(i)

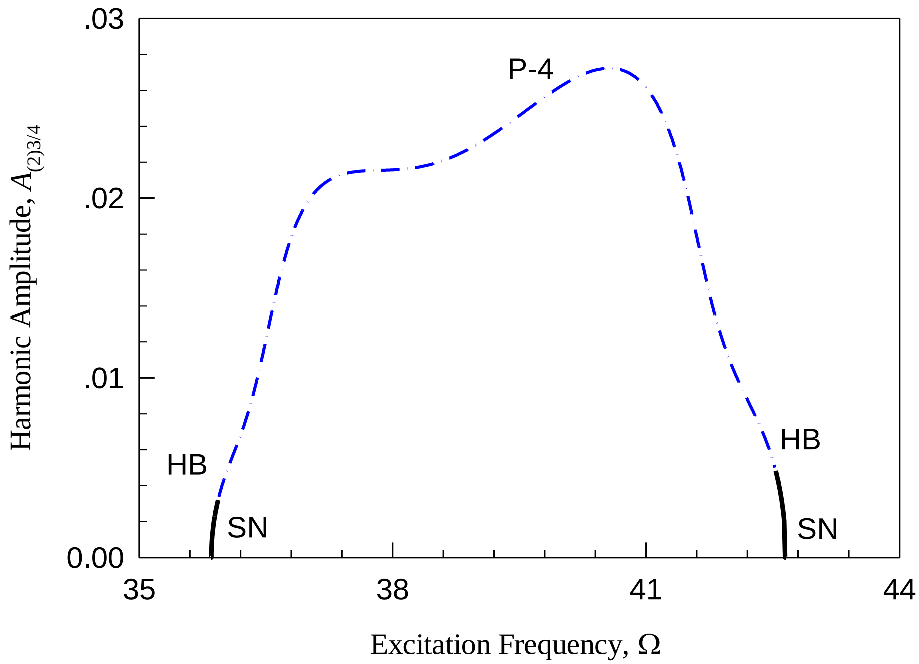


(ii)

Figure 5.2: Bifurcation tree from period-1 motion to chaos of the nonlinear cable structure in the torsional direction: frequency-amplitude curves of harmonic terms based on 30 harmonic terms (HB30): (i) $a_{20}^{(m)}$, (ii)-(xvi) $A_{(2)k/m}$ ($m=4, k=1,2,...,4; 8,12,16,20,24;116,117,118,119,120$), ($\zeta_y=0.0037, \zeta_\theta=0.0046, \eta_y=0.000922, \eta_\theta=0.0062, a_1=2.341, a_3=14.366, b_1=0.496,$ $b_3=1.265, U=6.77, \rho=1.255, d=33\times10^{-3}, k_y^{'}=5.0, k_\theta^{'}=2.0, Q_0=100$)

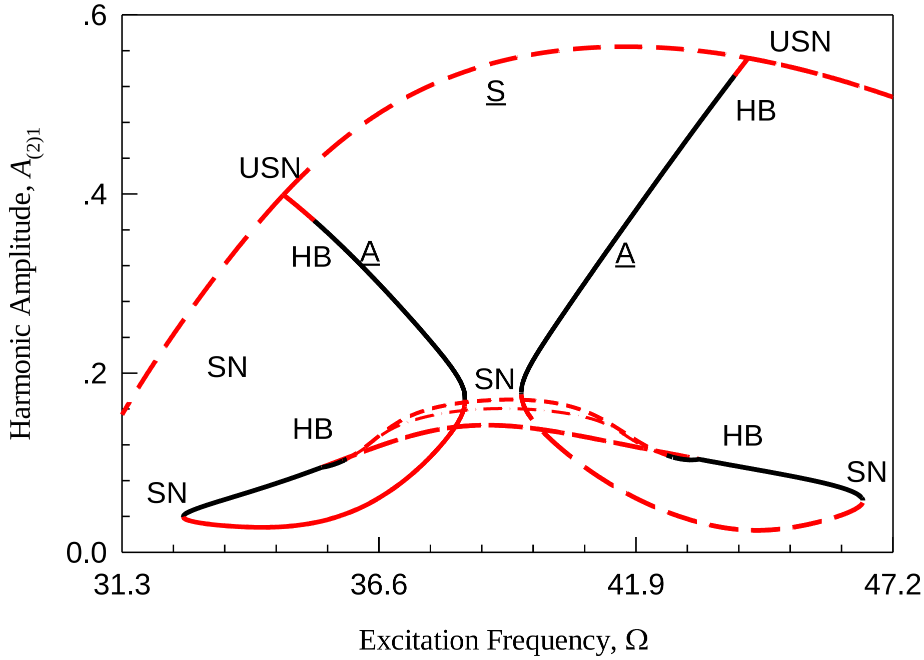


(iii)

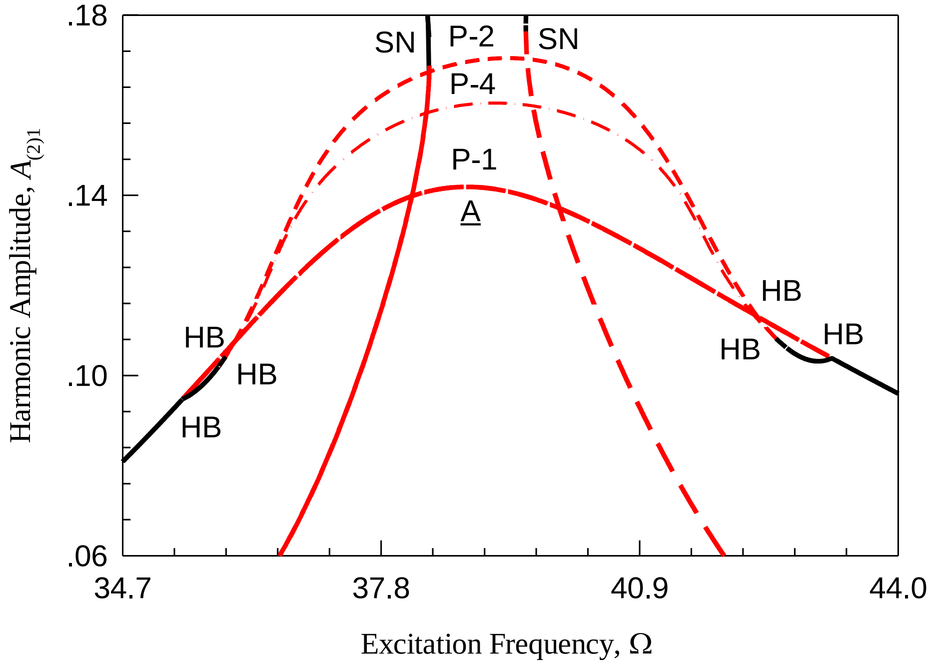


(iv)

Figure 5.2 Continued

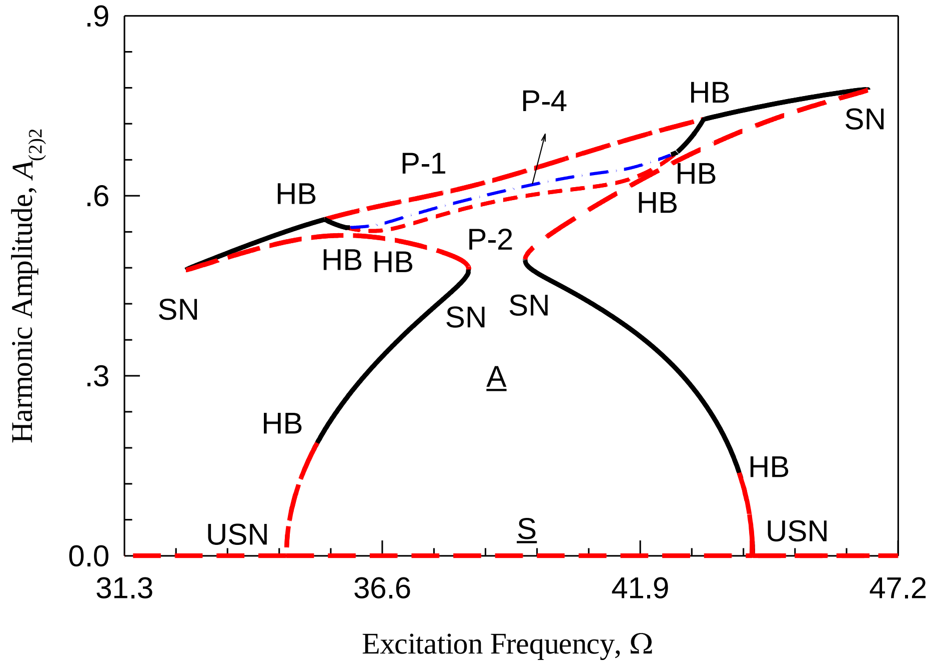


(v)

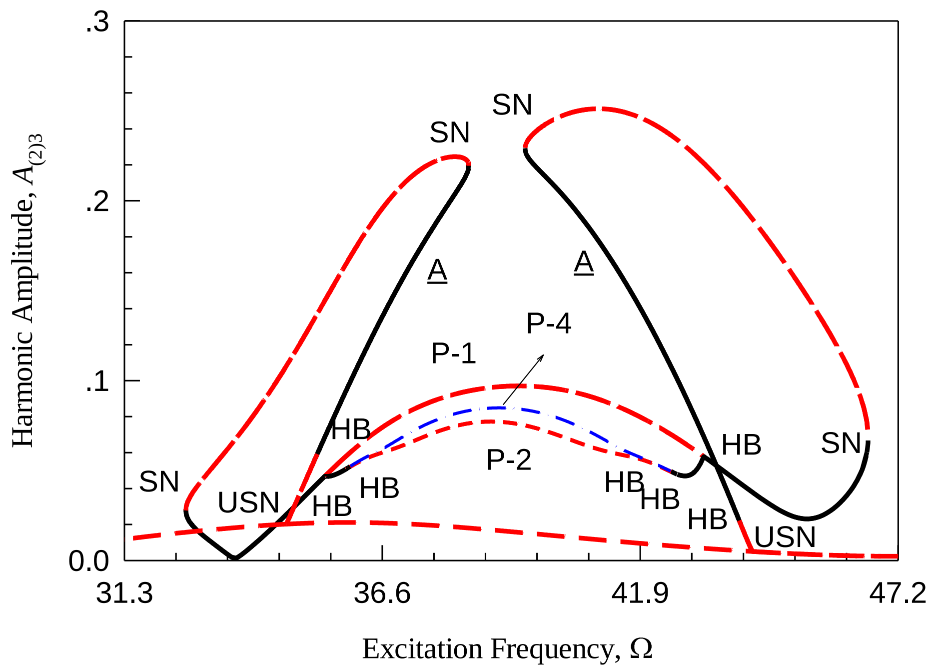


(vi)

Figure 5.2 Continued

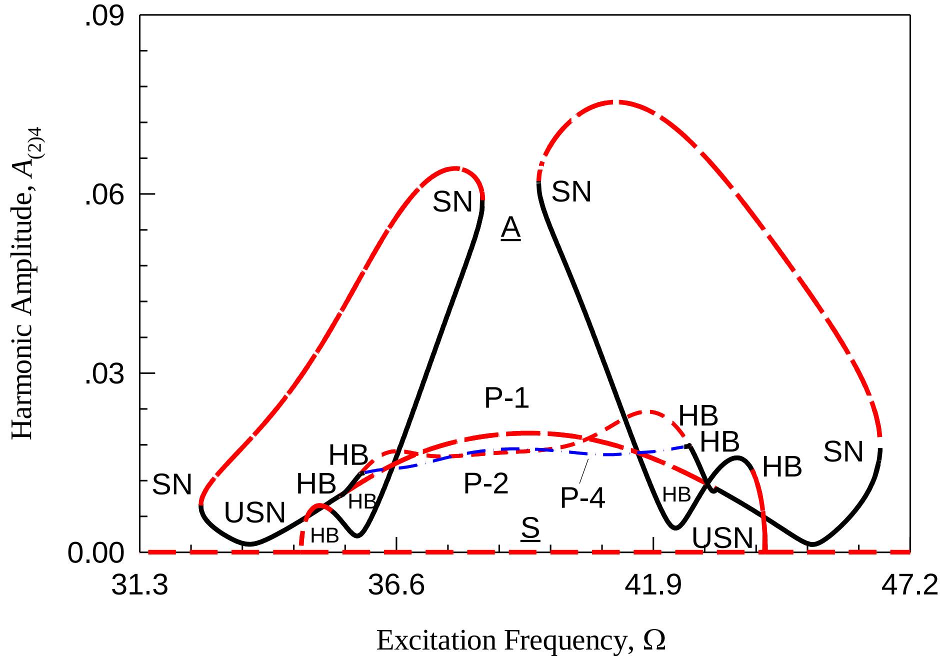


(vii)

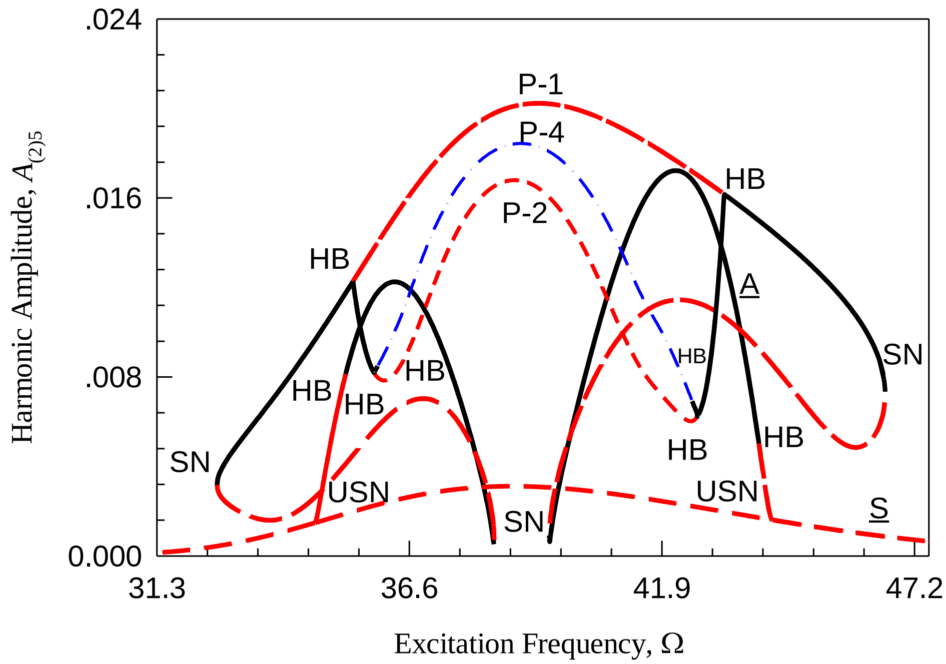


(viii)

Figure 5.2 Continued

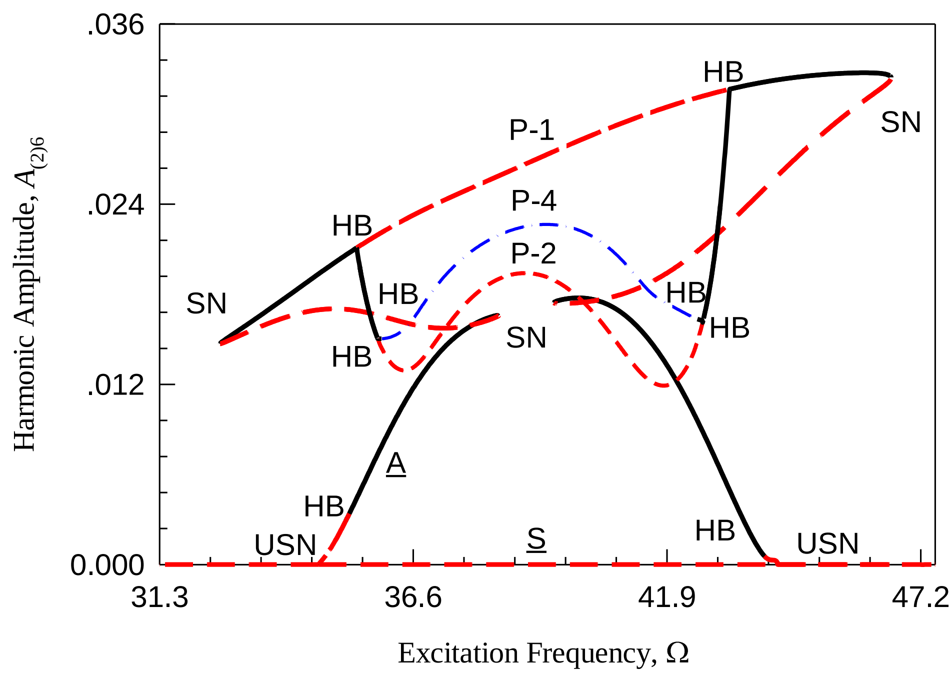


(ix)

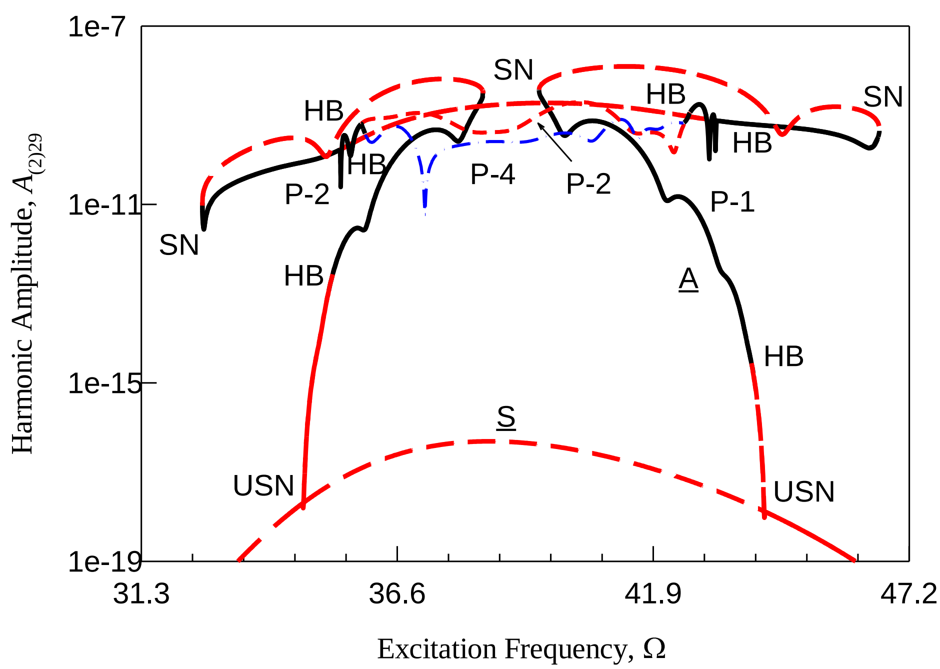


(x)

Figure 5.2 Continued

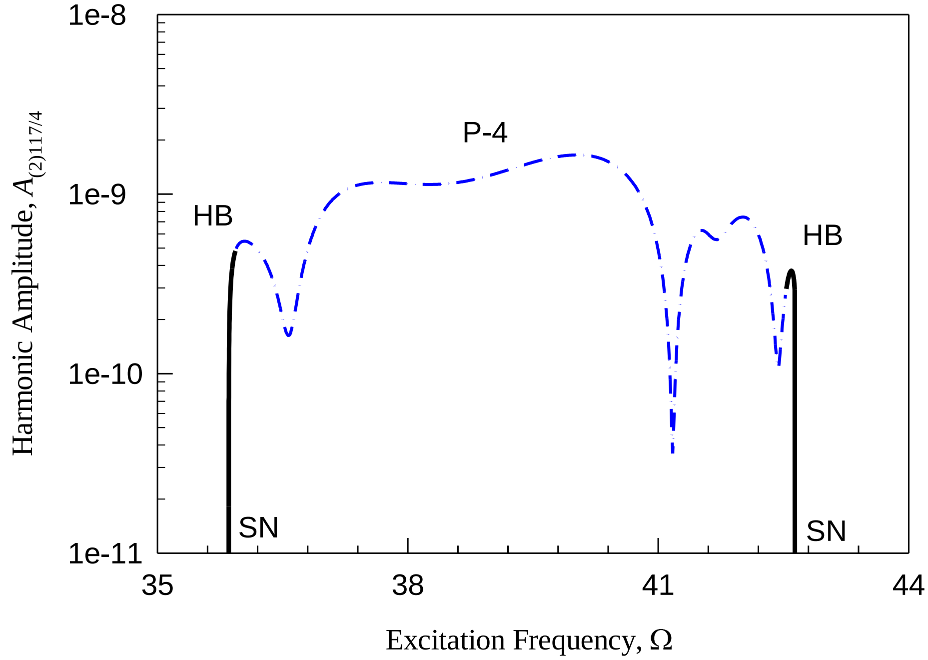


(xi)

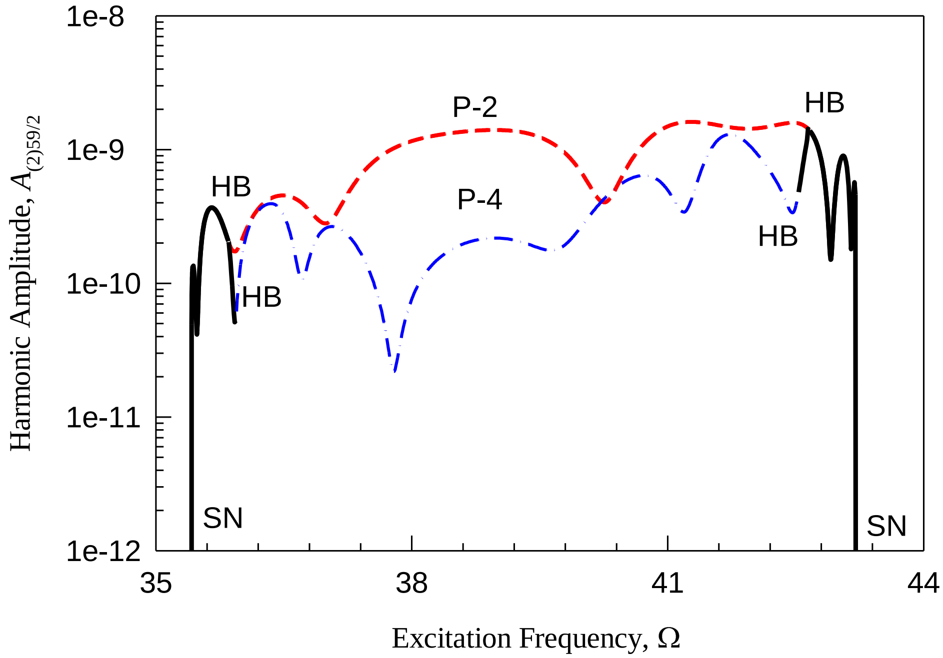


(xii)

Figure 5.2 Continued

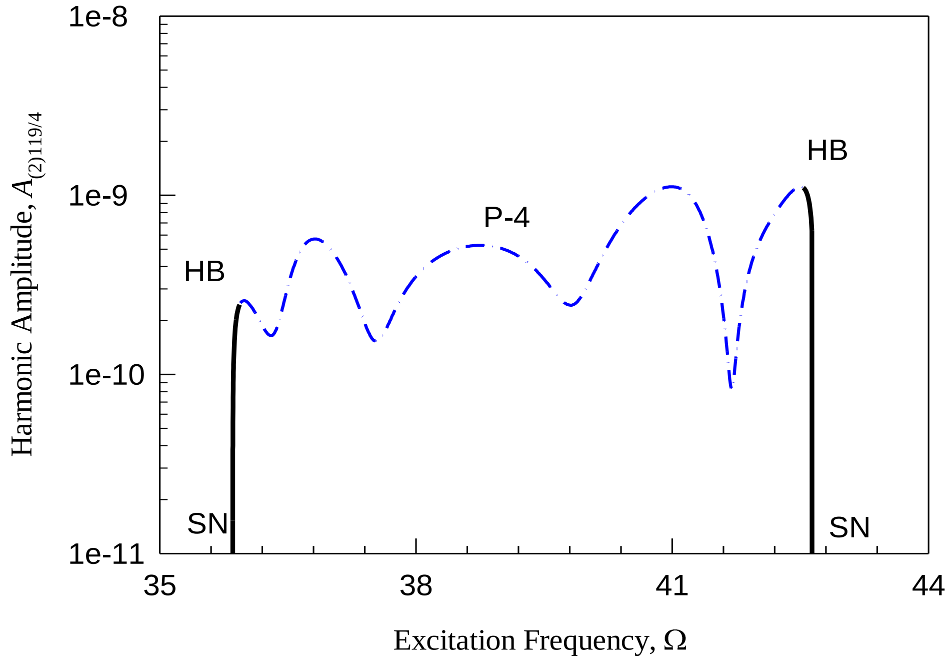


(xiii)

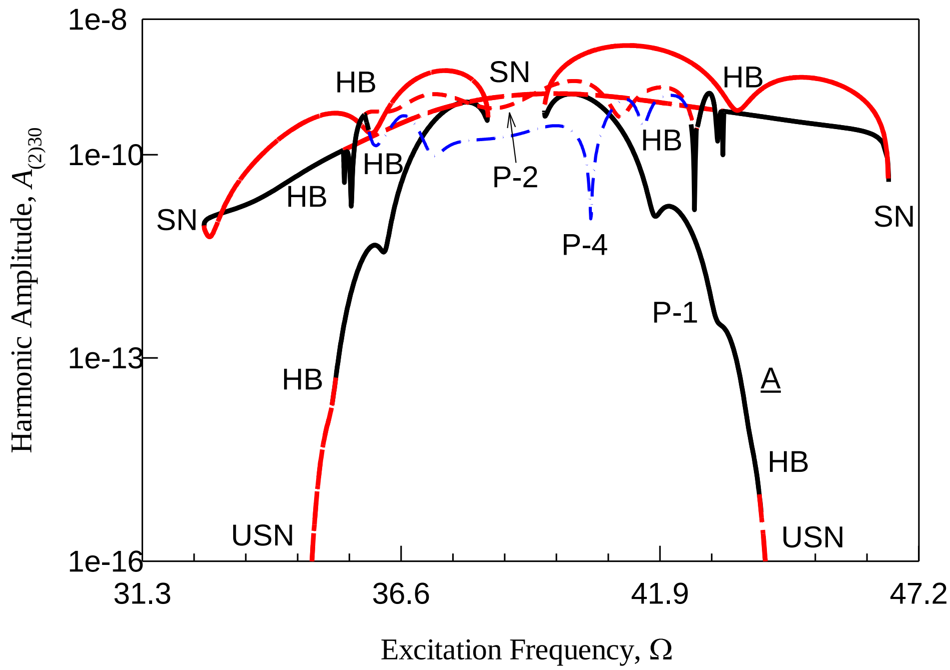


(xiv)

Figure 5.2 Continued



(xv)



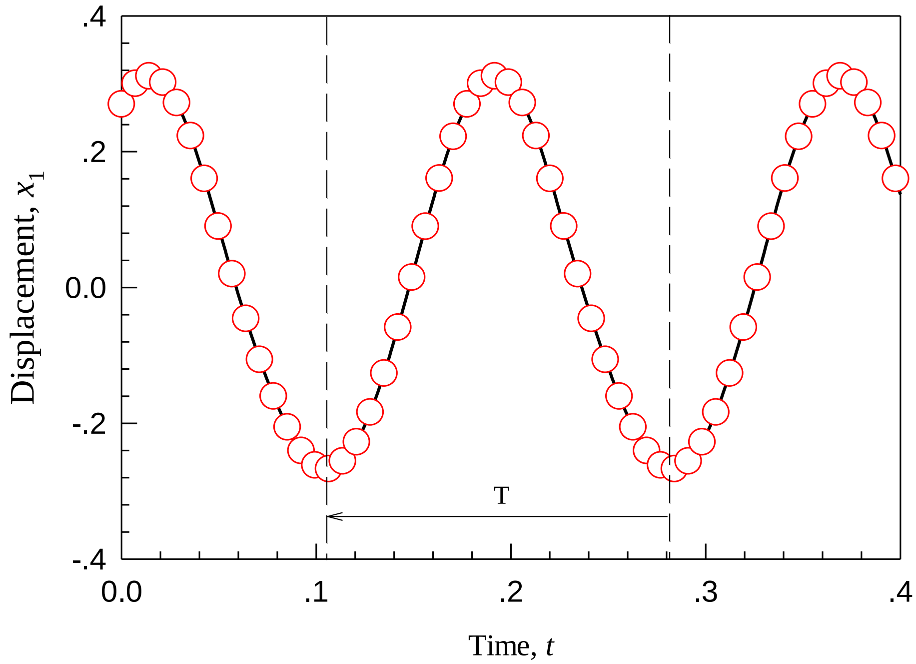
(xvi)

Figure 5.2 Continued

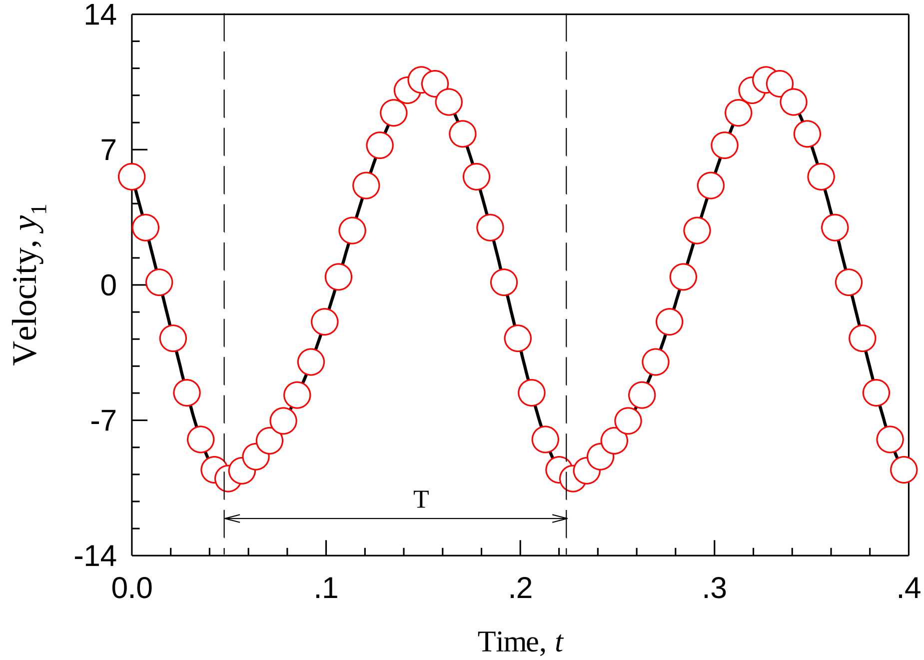
5.3 Numerical Illustrations

To illustrate periodic vibrations in such a nonlinear cable system, numerical and analytical solutions will be presented. The initial conditions for numerical simulations are computed from approximate analytical solutions of periodic solutions. In all plots, circular symbols give approximate solutions, and solid curves represent numerical simulation results. The acronym "IC" with a large circular symbol represents initial condition for all plots. The numerical solutions of periodic motions are generated via the mid-point scheme.

In Figure 5.3, a period-1 motion based on 30 harmonic terms (HB30) is presented for $\Omega = 35.4$ with other parameters in Eq.(47). The initial conditions are $(x_{10}, y_{10}) \approx$ (.27009549, 5.57673150) and $(x_{20}, y_{20}) =$ (-.44091253,-31.14258909). The displacement and velocity responses of the nonlinear cable in the transverse direction are presented in Fig.5.3 (i) and (ii), respectively. One period (T) for the period-1 motion is labeled. The trajectory is presented for over 40 periods in Figure 5.3 (iii). The initial condition is marked by a large circular symbol and labeled by "IC". For better understanding of harmonic contributions, the harmonic amplitude spectrum of transverse displacement is presented in Figure 5.3 (iv). The harmonic amplitude spectrum is computed from the analytical solution. The main harmonic amplitudes are $a_{10} = .0120$, $A_{(1)1} \approx 0.2880$, $A_{(1)2} \approx 0.0128$, $A_{(1)3} \approx 2.8335\times10^{-3}$, $A_{(1)4} \approx 1.1036\times10^{-3}$, $A_{(1)5} \approx 3.4263\times10^{-4}$, $A_{(1)6} \approx 1.1193\times10^{-4}$, $A_{(1)7} \approx 3.3793\times10^{-5}$, $A_{(1)8} \approx 1.1175\times10^{-5}$, $A_{(1)9} \approx 6.8196\times10^{-6}$, $A_{(1)10} \approx 3.1456\times10^{-6}$, $A_{(1)11} \approx 1.2299\times10^{-6}$, $A_{(1)12} \approx 4.1251\times10^{-7}$, $A_{(1)13} \approx 2.3204\times10^{-7}$, $A_{(1)14} \approx 1.2723\times10^{-7}$, $A_{(1)15} \approx 5.5298\times10^{-8}$, $A_{(1)16} \approx 1.8668\times10^{-8}$. The other harmonic amplitudes of the transverse displacement are $A_{(1)k} \in (10^{-13}, 10^{-9})$ ($k = 17, 18, 19, \cdots, 30$) and $A_{(1)30} \approx 6.0428\times10^{-13}$. Meanwhile the displacement and velocity of the torsional motion are presented in Figure 5.3 (v) and (vi), respectively. The displacements and velocities in the transverse and torsional directions are very different. Thus, the trajectories in each direction are different. The trajectory in the torsional direction is presented in Figure 5.3 (vii), which is different from the trajectory in transverse direction. The harmonic amplitude spectrum of the torsional motion is presented in Figure 5.3 (viii) for effects of the harmonic amplitudes on the period-1 motions. The main harmonic amplitudes of the torsional motion are $a_{20} = 0.0268$, $A_{(2)1} \approx 0.0944$, $A_{(2)2} \approx 0.5603$, $A_{(2)3} \approx 0.463$, $A_{(2)4} \approx 9.2963\times10^{-3}$, $A_{(2)5} \approx 0.0122$, $A_{(2)6} \approx 0.0211$, $A_{(2)7} \approx 3.9739\times10^{-3}$, $A_{(2)8} \approx 8.9740\times10^{-4}$, $A_{(2)9} \approx 7.4163\times10^{-4}$, $A_{(2)10} \approx 8.0149\times10^{-4}$, $A_{(2)11} \approx 2.4159\times10^{-4}$, $A_{(2)12} \approx 6.1794\times10^{-5}$, $A_{(2)13} \approx 6.1794\times10^{-5}$, $A_{(2)14} \approx 3.1855\times10^{-5}$, $A_{(2)15} \approx 1.2881\times10^{-5}$, $A_{(2)16} \approx 3.7016\times10^{-6}$ and $A_{(2)17} \approx 1.5573\times10^{-6}$. The other harmonic amplitudes of the torsional motion are $A_{(2)k} \in (10^{-10}, 10^{-7})$ $(k = 18, 19, \cdots, 30)$ and $A_{(2)30} \approx 1.1469\times10^{-10}$. Since the period-1 motion possesses a very large excitation frequency, the 30 harmonic terms can give a very accurate analytical solution.

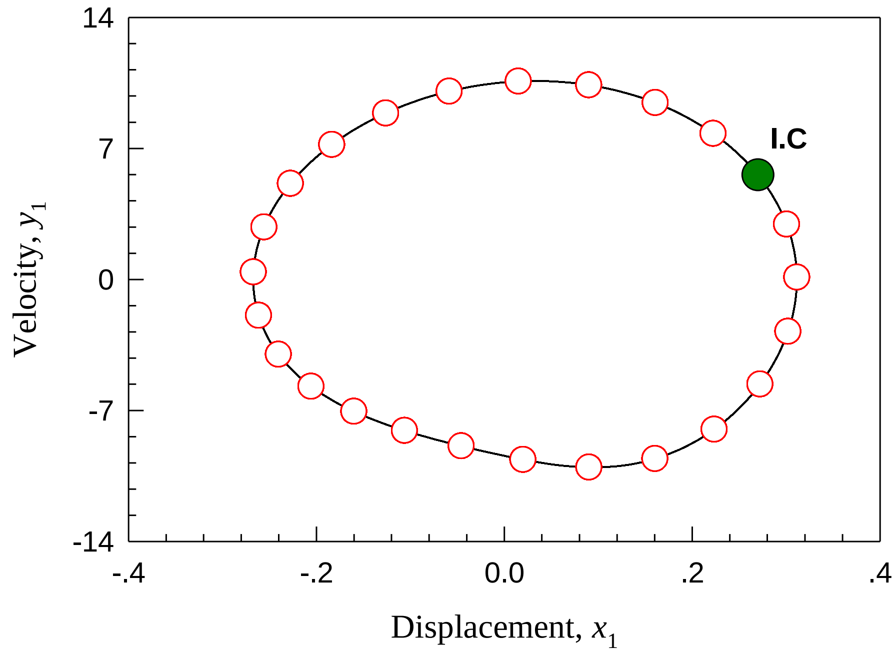


(i)

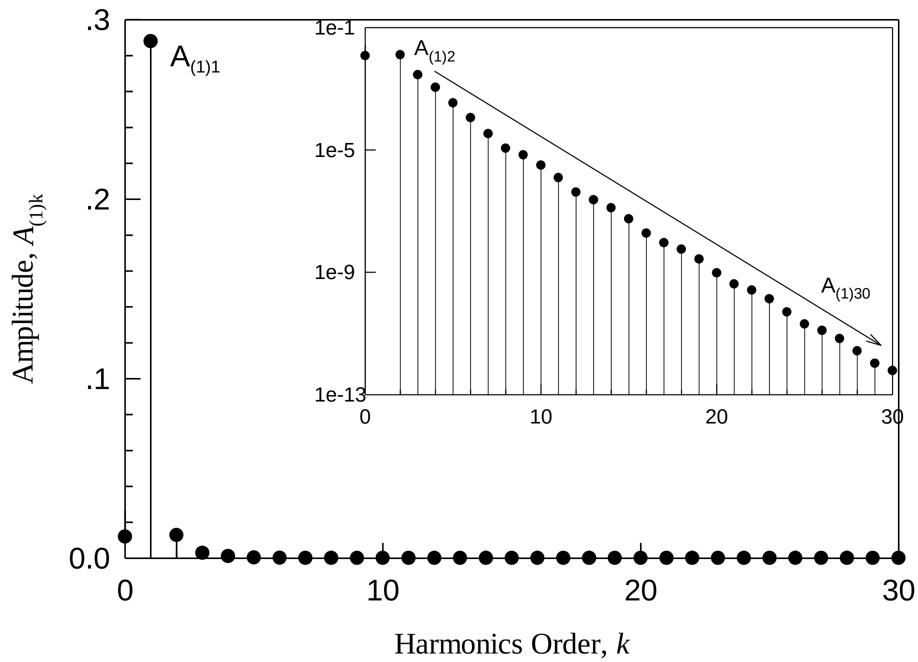


(ii)

Figure 5.3: Stable period-1 motion of nonlinear cable structure in transverse direction ($\Omega = 35.4$, HB30): (i) displacement x_1, (ii) velocity y_1; (iii) trajectory (x_1, y_1), (iv) harmonic amplitudes $A_{(1)k}$ $(k = 1, 2, \cdots, 30)$. Motion in torsional direction: (v) displacement x_2, (vi) velocity y_2; (vii) trajectory (x_2, y_2), (viii) harmonic amplitudes $A_{(2)k}$ $(k = 1, 2, \cdots, 30)$. Initial conditions $(x_{10}, y_{10}) \approx (.27009549, 5.57673150)$ and $(x_{20}, y_{20}) \approx (-.44091253, -31.14258909)$.

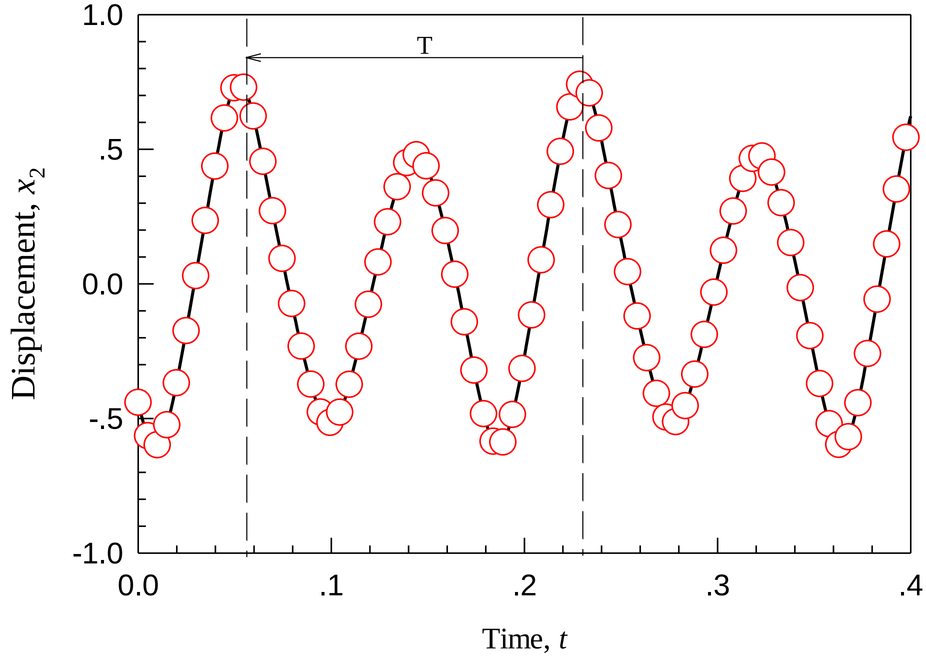


(iii)

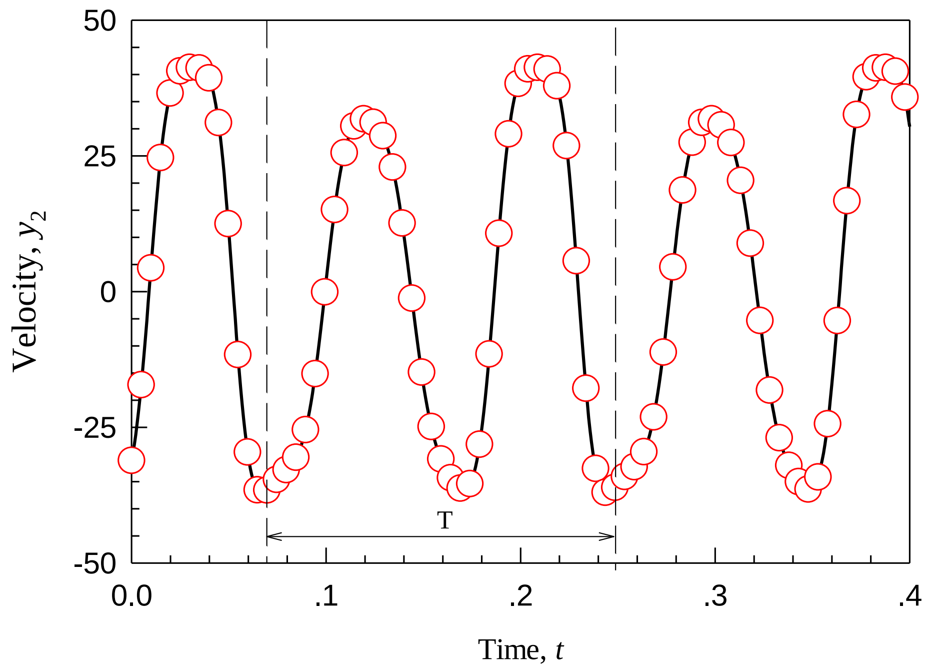


(iv)

Figure 5.3 Continued

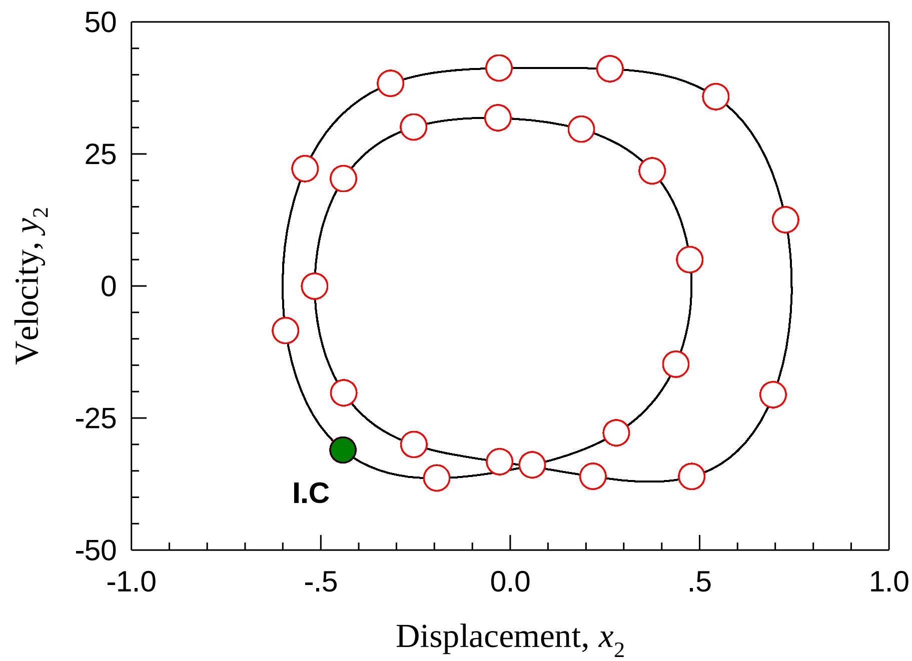


(v)

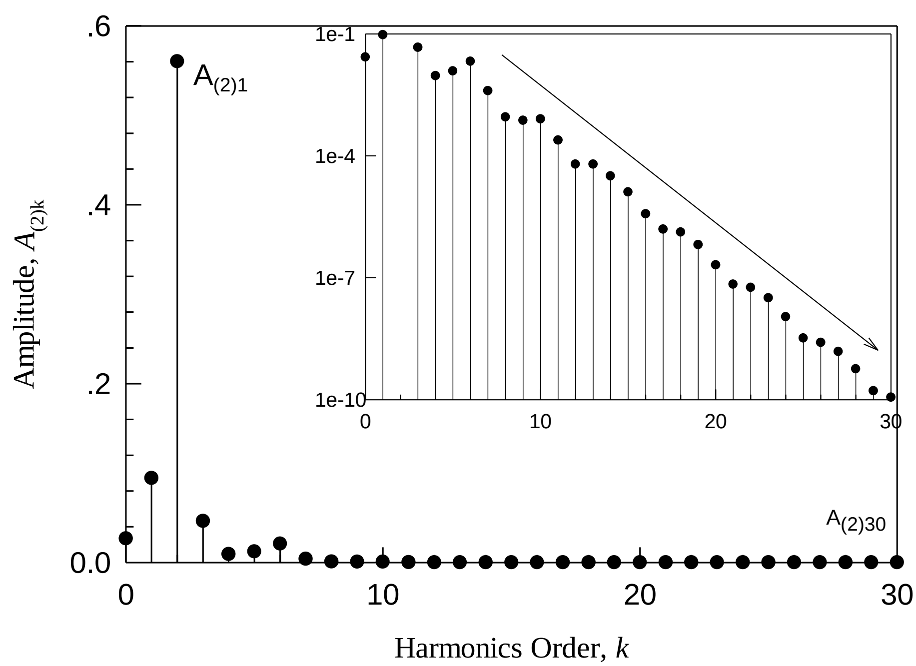


(vi)

Figure 5.3 Continued



(vii)



(viii)

Figure 5.3 Continued

On the same side of bifurcation tree of period-1 motion to chaos, consider a period-2 motion. Such a periodic motion is expressed analytically by 60 harmonic terms for $\Omega = 35.6$, as shown in Figure 5.4. With other parameters in Eq.(47), the analytical solution gives the initial condition $(x_{10}, y_{10}) = (.267991, 5.835643)$ and $(x_{20}, y_{20}) = (-.509682, -33.127622)$, which is used for numerical simulation. The displacement and velocity responses in the transverse direction of such nonlinear cable system are presented in Figure 5.4 (i) and (ii), respectively. Two periods (2*T*) for the period-2 motion are labeled. The trajectory in the transverse direction is presented for over 40 periods in Figure 5.4 (iii). The initial condition is marked by a large circular symbol and labeled by "IC". Compared to one cycle of period-1 motion, two cycles are observed for the period-2 motion. To understand the difference between period-1 and period-2 motions, the harmonic amplitude spectrum of the transverse motion in the perod-2 motion is presented. In Figure 5.4 (iv), the harmonic amplitude spectrum is computed from analytical solutions. The main harmonic amplitudes of the transverse motion for the period-2 motion are

$a_{10}^{(2)} = 0.0121,\ \ A_{(1)1/2} \approx 2.8461\times10^{-3},\ A_{(1)1} \approx 0.2929,\ A_{(1)3/2} \approx 3.7998\times10^{-3},$

$A_{(1)2} \approx 0.0130,\ A_{(1)5/2} \approx 1.4045\times10^{-3},\ \ A_{(1)3} \approx 2.8057\times10^{-3},\ \ A_{(1)7/2} \approx 4.4562\times10^{-4},$

$A_{(1)4} \approx 1.0677\times10^{-3},\ A_{(1)9/2} \approx 2.3821\times10^{-4},\ \ A_{(1)5} \approx 3.1912\times10^{-4},\ \ A_{(1)11/2} \approx 1.1687\times10^{-4},$

$A_{(1)6} \approx 9.6839\times10^{-5},\ A_{(1)13/2} \approx 4.9813\times10^{-5},\ \ A_{(1)7} \approx 2.5303\times10^{-5},\ \ A_{(1)15/2} \approx 1.9771\times10^{-5},$

$A_{(1)8} \approx 4.2153\times10^{-6},\ \ A_{(1)17/2} \approx 8.7984\times10^{-6},\ \ A_{(1)9} \approx 3.1574\times10^{-6},\ \ A_{(1)15/2} \approx 4.8856\times10^{-6},$

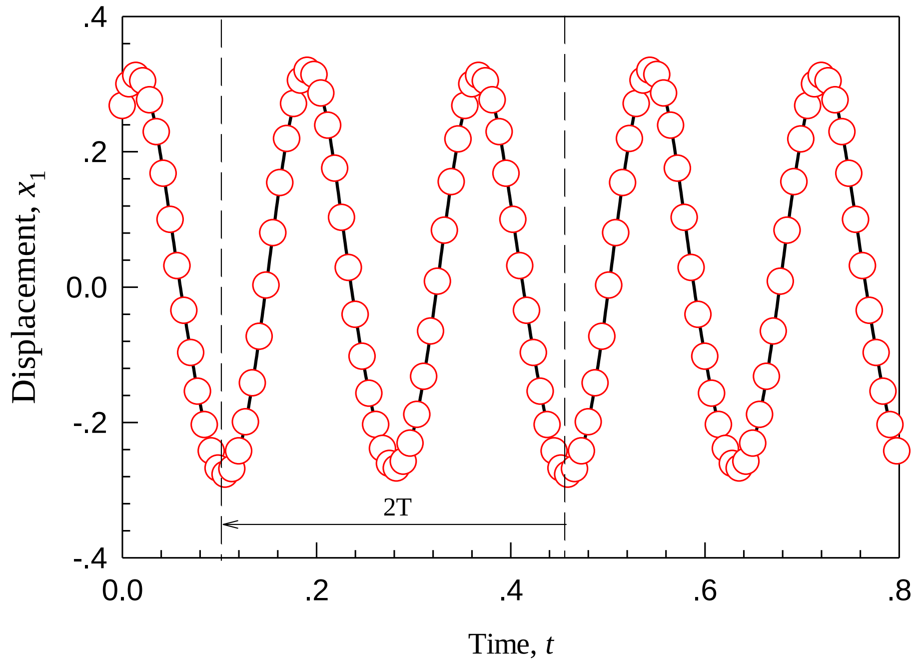
$A_{(1)10} \approx 1.0428\times10^{-6},\ \ \ A_{(1)21/2} \approx 2.2839\times10^{-6},\ \ A_{(1)11} \approx 3.8132\times10^{-7},\ \ A_{(1)23/2} \approx 9.1098\times10^{-7},$

$A_{(1)12} \approx 2.0320\times10^{-7},\ \ A_{(1)25/2} \approx 3.3349\times10^{-7},\ \ A_{(1)13} \approx 1.1084\times10^{-7},\ \ A_{(1)27/2} \approx 1.7755\times10^{-7},$

$A_{(1)14} \approx 3.5176\times10^{-8},\ \ A_{(1)29/2} \approx 9.1208\times10^{-8},\ \ A_{(1)15} \approx 3.5296\times10^{-8},\ \ A_{(1)31/2} \approx 3.5687\times10^{-8},$

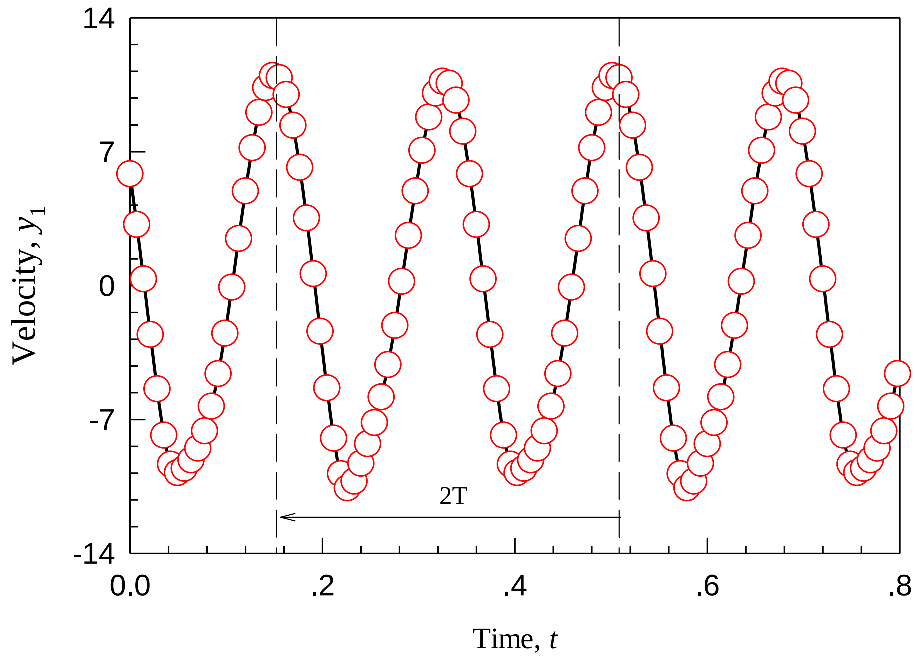
$A_{(1)16} \approx 2.3985\times10^{-8}$ $A_{(1)33/2} \approx 9.0104\times10^{-9}$, and $A_{(1)17} \approx 1.1034\times10^{-8}$. The other harmonic amplitudes of the transverse motion are $A_{(1)k/2} \in (10^{-14},10^{-9})$ ($k = 35,36,\cdots,60$) and $A_{(1)30} \approx 3.0704\times10^{-14}$. The amplitude drops exponentially as the harmonic order k increases. So the ordinate is in linear logarithm scale. In such nonlinear cable system, the displacement and velocity in the torsional direction are presented in Figure 5.4 (v) and (vi), respectively. The trajectory of in the torsional direction is presented in Figure 5.4 (vii). The number of cycles of the trajectory doubles for the period-2 motion compared with period-1 motion, which cannot be obtained from the traditional analytical methods. The motion in both direction are not similar. Both displacement and velocity in the torsional direction are greater than the transverse direction for such parameter set. The harmonic amplitude spectrum of the torsional motion is presented in Figure 5.4 (viii) for effects of the harmonic amplitudes on the period-2 motions. The main harmonic amplitudes of the torsional motion are $a_{20}^{(2)} = 0.0251$, $A_{(2)1/2} \approx 0.0113$, $A_{(2)1} \approx 0.0969$, $A_{(2)3/2} \approx 0.0403$, $A_{(2)2} \approx 0.5537$, $A_{(2)5/2} \approx 0.0949$, $A_{(2)3} \approx 0.0475$, $A_{(2)7/2} \approx 0.0187$, $A_{(2)4} \approx 0.0106$, $A_{(2)9/2} \approx 7.9754\times10^{-3}$, $A_{(2)5} \approx 9.8917\times10^{-3}$, $A_{(2)11/2} \approx 9.0746\times10^{-3}$, $A_{(2)6} \approx 0.0179$, $A_{(2)13/2} \approx 9.4428\times10^{-3}$, $A_{(2)7} \approx 2.7778\times10^{-3}$, $A_{(2)15/2} \approx 2.4192\times10^{-3}$, $A_{(2)8} \approx 7.4625\times10^{-4}$, $A_{(2)17/2} \approx 7.7090\times10^{-4}$, $A_{(2)9} \approx 3.8926\times10^{-4}$, $A_{(2)17/2} \approx 6.6504\times10^{-4}$, $A_{(2)10} \approx 4.3020\times10^{-4}$, $A_{(2)21/2} \approx 5.2707\times10^{-4}$, $A_{(2)11} \approx 1.3793\times10^{-4}$, $A_{(2)23/2} \approx 1.6804\times10^{-4}$, $A_{(1)12} \approx 6.7403\times10^{-5}$, $A_{(2)25/2} \approx 4.3338\times10^{-5}$, $A_{(2)13} \approx 3.0583\times10^{-5}$, $A_{(2)27/2} \approx 3.0807\times10^{-5}$, $A_{(2)14} \approx 1.0253\times10^{-5}$, $A_{(2)29/2} \approx 2.2424\times10^{-5}$, $A_{(2)15} \approx 9.1008\times10^{-6}$, $A_{(2)31/2} \approx 8.2094\times10^{-6}$, $A_{(2)16} \approx 5.6049\times10^{-6}$, $A_{(2)33/2} \approx 1.5257\times10^{-6}$, and $A_{(2)17} \approx 2.5587\times10^{-6}$. The other harmonic amplitudes of the torsional motion are $A_{(1)k/2} \in (10^{-11},10^{-7})$ ($k = 35,36,\cdots,60$)

and $A_{(2)30} \approx 4.3065 \times 10^{-11}$. From the numerical illustration, the 60 harmonic terms can give an accurate analytical solution of period-2 motion in the vicinity of $\Omega = 35.6,$. That is, the accuracy of the analytical solution is about 10^{-11}. The tradition perturbation method with only few terms cannot achieve such an accurate solution.

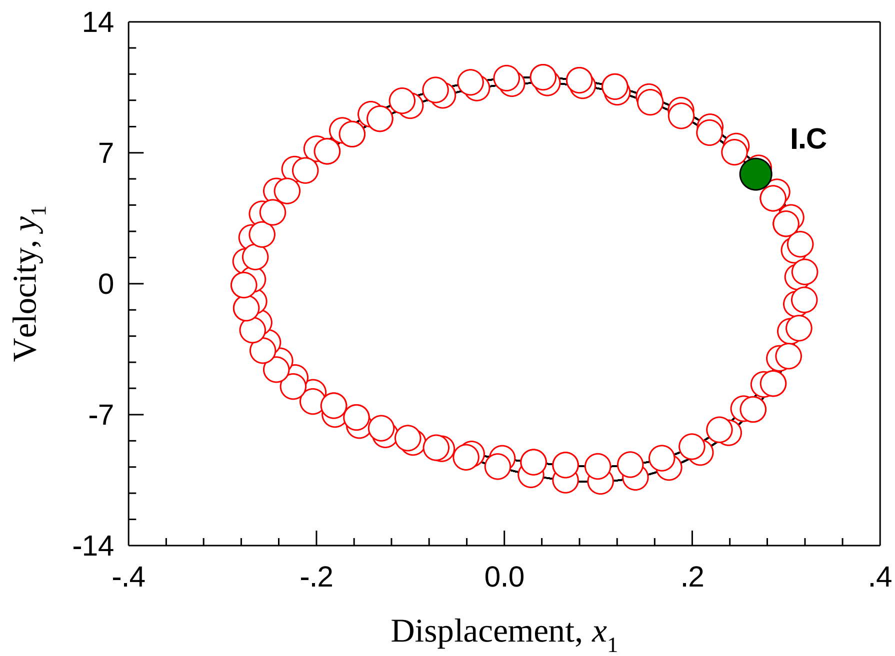


(i)

Figure 5.4: Stable period-2 motion of nonlinear cable structure in transverse direction ($\Omega = 35.6$, HB60): (i) displacement x_1, (ii) velocity y_1; (iii) trajectory (x_1, y_1), (iv) harmonic amplitudes $A_{(1)k/2}$ ($k = 1, 2, \cdots, 60$) . Motion in torsional direction: (v) displacement x_2, (vi) velocity y_2; (vii) trajectory (x_2, y_2), (viii) harmonic amplitudes $A_{(2)k/2}$ ($k = 1, 2, \cdots, 60$). Initial conditions $(x_{10}, y_{10}) = (.267991, 5.835643)$ and $(x_{20}, y_{20}) = (-.509682, -33.127622)$.

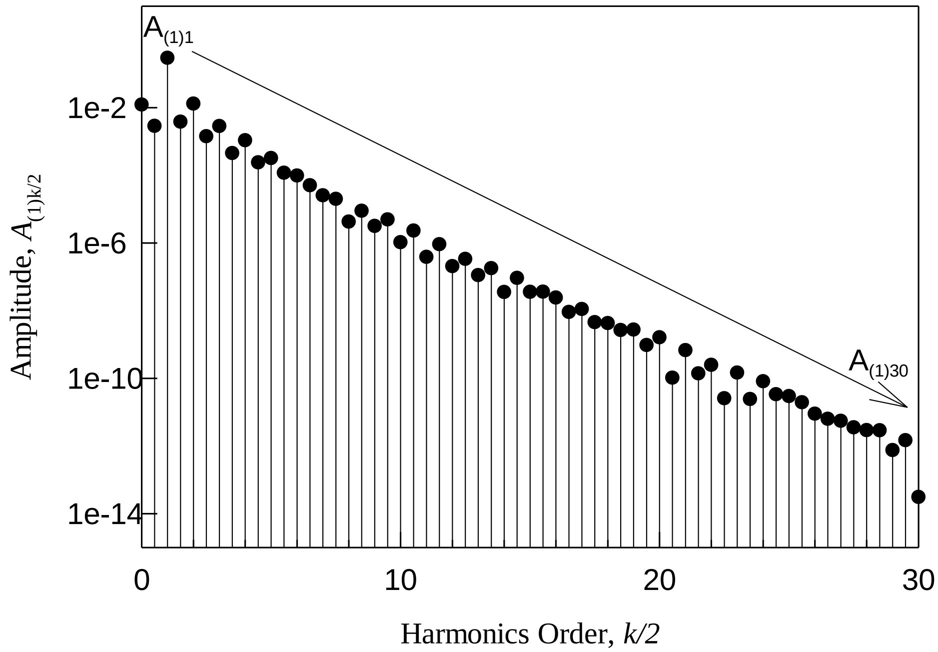


(ii)

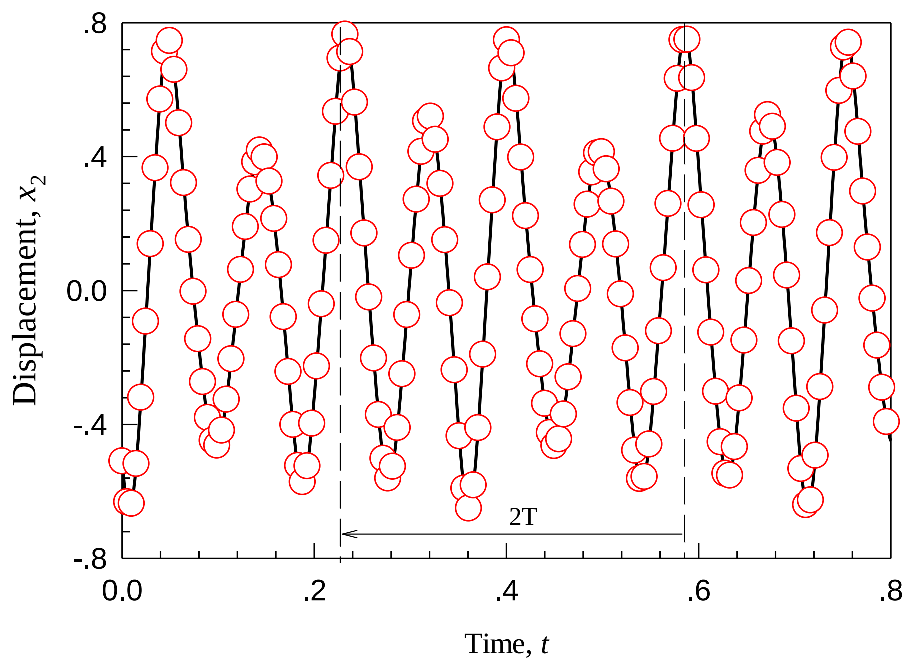


(iii)

Figure 5.4 Continued

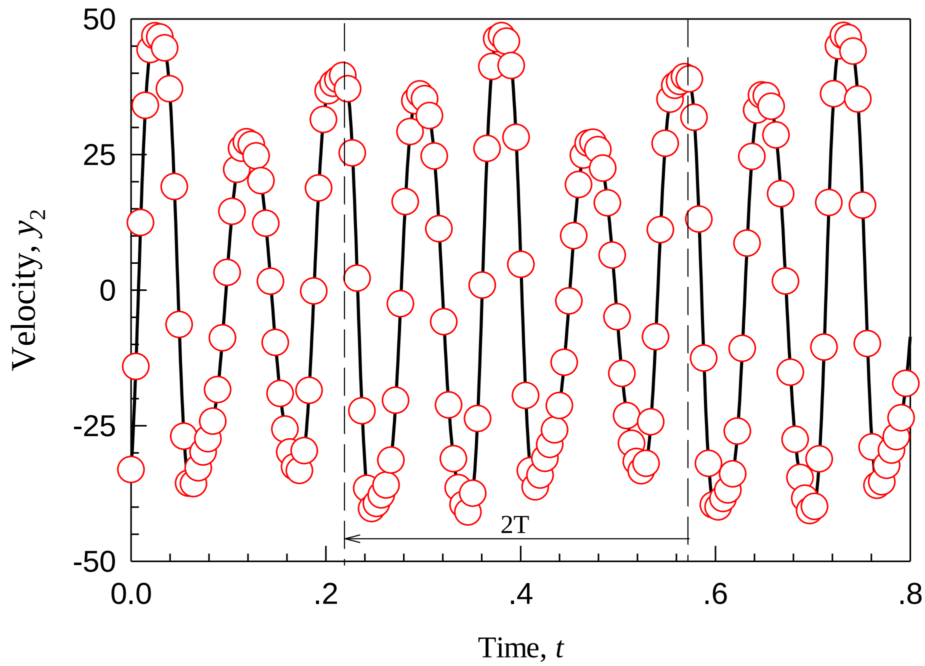


(iv)

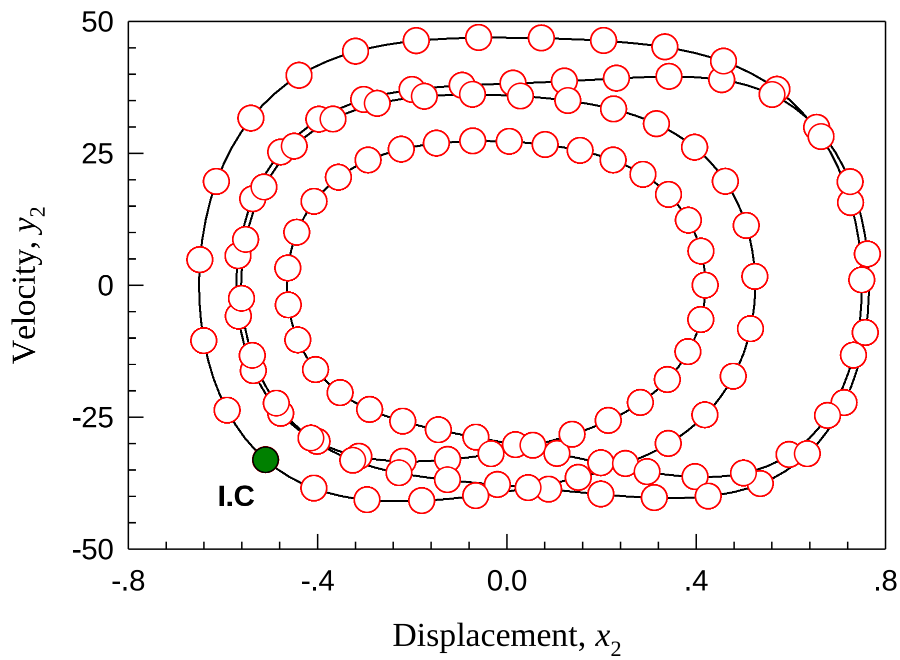


(v)

Figure 5.4 Continued

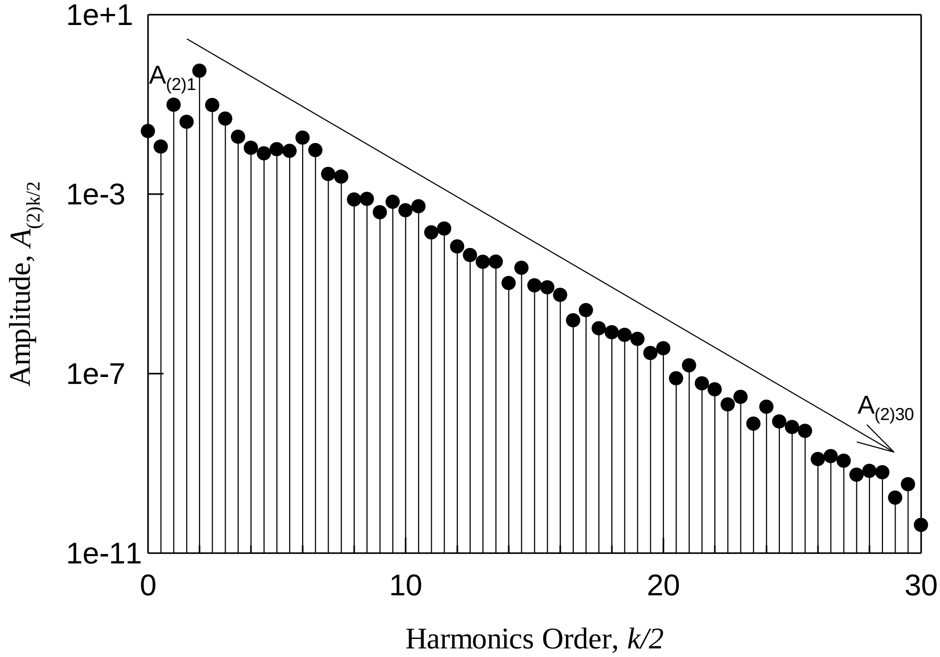


(vi)



(vii)

Figure 5.4 Continued



(viii)

Figure 5.4 Continued

To future demonstrate the trend from periodic motion to chaos, consider a period-4 motion on the same branch. Such a periodic motion is expressed analytically by 120 harmonic terms for $\Omega = 38.56$ as shown in Figure 5.5. With other parameters in Eq.(47), the analytical solution gives the initial condition $(x_{10}, y_{10}) = (.218801, 5.424894)$ and $(x_{20}, y_{20}) = (.253783,$ $36.973190),$ which is used for numerical simulation. The displacement and velocity responses in the transverse direction of such nonlinear cable system are presented in Figure 5.5 (i) and (ii), respectively. Four periods (4*T*) for the period-4 motion are labeled. The trajectory in the transverse direction is presented for over 80 periods in Figure 5.5 (iii). The initial condition is marked by a large circular symbol and labeled by "IC". Compared to one cycle of period-2 motion, Four cycles are observed for the period-4 motion. To understand the difference between period-2 and period-4 motions, the harmonic amplitude spectrum of the transverse motion in the perod-4 motion is presented. In Figure 5.5 (iv), the harmonic amplitude spectrum is computed from analytical solutions. The main harmonic amplitudes of the transverse motion for the period-4 motion are $a_{10}^{(4)} = $-.01235, $A_{(1)1/4} \approx 2.0805\times10^{-4}$, $A_{(1)1/2} \approx 4.4244\times10^{-3}$, $A_{(1)3/4} \approx 2.0062\times10^{-4}$, $A_{(1)1} \approx .2989$, $A_{(1)5/4} \approx 1.3162\times10^{-4}$, $A_{(1)3/2} \approx 5.1751\times10^{-3}$, $A_{(1)7/4} \approx 2.2084\times10^{-4}$, $A_{(1)2} \approx .0132$, $A_{(1)9/4} \approx 8.9785\times10^{-5}$, $A_{(1)5/2} \approx 2.0390\times10^{-3}$, $A_{(1)11/4} \approx 6.6322\times10^{-5}$, $A_{(1)3} \approx 2.8046\times10^{-3}$, $A_{(1)13/4} \approx 1.7694\times10^{-5}$, $A_{(1)7/2} \approx 6.3457\times10^{-4}$, $A_{(1)15/4} \approx 2.8438\times10^{-5}$, $A_{(1)4} \approx 1.0382\times10^{-3,}$ $A_{(1)17/4} \approx 7.7480\times10^{-6}$, $A_{(1)9/2} \approx 3.3609\times10^{-4,}$ $A_{(1)19/4} \approx 1.4134\times10^{-5}$, $A_{(1)5} \approx 3.0028\times10^{-4}$, $A_{(1)21/4} \approx 4.0164\times10^{-6}$ $A_{(1)11/2} \approx 1.6253\times10^{-4}$, $A_{(1)23/4} \approx 7.5843\times10^{-6}$, $A_{(1)6} \approx 8.5266\times10^{-5}$, $A_{(1)25/4} \approx 2.6506\times10^{-6}$, $A_{(1)13/2} \approx 6.8390\times10^{-5}$, $A_{(1)27/4} \approx 3.4733\times10^{-6}$, $A_{(1)7} \approx 2.1199\times10^{-5}$, $A_{(1)29/4} \approx 1.6757\times10^{-6}$, $A_{(1)15/2} \approx 2.5937\times10^{-5}$, $A_{(1)31/4} \approx 1.2687\times10^{-6}$, $A_{(1)8} \approx 3.9352\times10^{-6}$ $A_{(1)33/4} \approx 7.3459\times10^{-7}$, $A_{(1)17/2} \approx 1.0324\times10^{-5}$, $A_{(1)35/4} \approx 4.9254\times10^{-7}$, $A_{(1)9} \approx 1.9887\times10^{-6}$ $A_{(1)37/4} \approx 3.6490\times10^{-7}$, $A_{(1)19/2} \approx 5.5842\times10^{-6}$, $A_{(1)39/4} \approx 2.8922\times10^{-7}$, $A_{(1)10} \approx 1.0619\times10^{-6}$. The other harmonic amplitudes in the transverse direction are $A_{(1)k/4} \in (10^{-13}, 10^{-7})$ ($k = 41, 42, \cdots, 120$) and $A_{(1)30} \approx 1.6439\times10^{-12}$. The amplitude drops exponentially as the harmonic order k increases. So the ordinate is in linear logarithm scale. In such nonlinear cable system, the displacement and velocity in the torsional direction are presented in Figure 5.5 (v) and (vi), respectively. The trajectory of in the torsional direction is presented in Figure 5.5 (vii). The number of cycles of the trajectory doubles again for the period-4 motion compared with period-2 motion, which cannot be obtained from the traditional analytical methods. The motion in both direction are not similar. Both displacement and velocity in the torsional direction is greater than the transverse direction for such parameter set. The harmonic amplitude spectrum of the torsional motion is presented in Figure 5.5 (viii) for effects of the harmonic amplitudes on the period-4 motions. The main harmonic amplitudes of the torsional motion are are $a_{20}^{(4)} = -.5866$,

$A_{(2)1/4} \approx 6.8405\times10^{-4}$, $A_{(2)1/2} \approx 0.0186$, $A_{(2)3/4} \approx 8.2784\times10^{-4}$, $A_{(2)1} \approx .1022$,

$A_{(2)5/4} \approx 2.7872\times10^{-3}$, $A_{(2)3/2} \approx 0.533$, $A_{(2)7/4} \approx 5.2675\times10^{-3}$, $A_{(2)2} \approx .5470$,

$A_{(2)9/4} \approx 5.7471\times10^{-3}$, $A_{(2)5/2} \approx 0.1349$, $A_{(2)11/4} \approx 4.3237\times10^{-3}$, $A_{(2)3} \approx 0.0509$,

$A_{(2)13/4} \approx 9.9988\times10^{-4}$, $A_{(2)7/2} \approx 0.0267$, $A_{(2)15/4} \approx 1.1368\times10^{-3}$, $A_{(2)4} \approx 0.0131$,

$A_{(2)17/4} \approx 5.4933\times10^{-4}$, $A_{(2)9/2} \approx 0.0108$, $A_{(2)19/4} \approx 5.4815\times10^{-4}$, $A_{(2)5} \approx 8.2161\times10^{-3}$,

$A_{(2)21/4} \approx 3.0590\times10^{-4}$ $A_{(2)11/2} \approx 0.0119$, $A_{(2)23/4} \approx 7.7866\times10^{-4}$, $A_{(2)6} \approx 0.0151$,

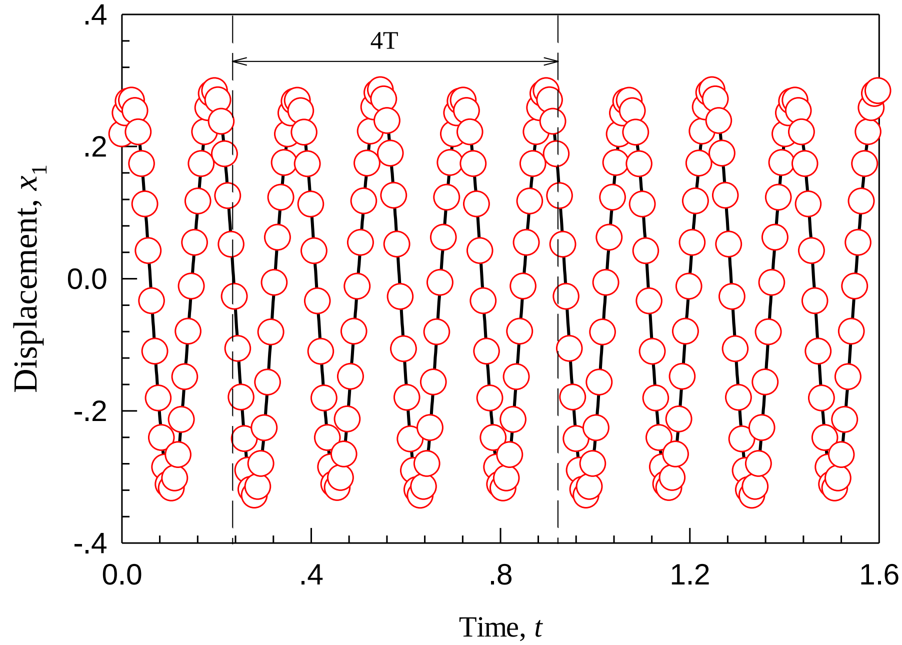
$A_{(2)25/4} \approx 3.1348\times10^{-4}$, $A_{(2)13/2} \approx 0.0125$, $A_{(2)27/4} \approx 5.8997\times10^{-4}$, $A_{(2)7} \approx 3.0959\times10^{-3}$,

$A_{(2)29/4} \approx 2.9353\times10^{-4}$, $A_{(2)15/2} \approx 3.0963\times10^{-5}$, $A_{(2)31/4} \approx 1.8732\times10^{-4}$, $A_{(2)8} \approx 1.3720\times10^{-3}$

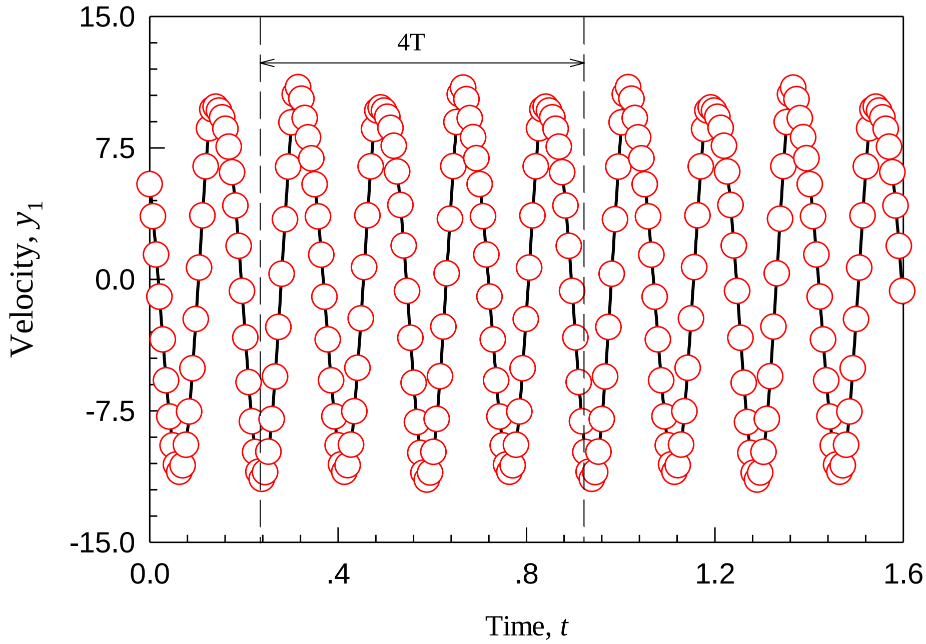
$A_{(2)33/4} \approx 1.3638\times10^{-4}$, $A_{(2)17/2} \approx 8.0817\times10^{-4}$, $A_{(2)35/4} \approx 4.4682\times10^{-5}$, $A_{(2)9} \approx 5.4228\times10^{-4}$

$A_{(2)37/4} \approx 5.3089\times10^{-5}$, $A_{(2)19/2} \approx 7.1048\times10^{-4}$, $A_{(2)39/4} \approx 3.2490\times10^{-5}$, $A_{(1)10} \approx 2.5271\times10^{-4}$.

The other harmonic amplitudes of the torsional motion are $A_{(1)k/2} \in (10^{-11}, 10^{-5})$ ($k = 41, 42, \cdots, 120$) and $A_{(2)30} \approx 3.7727\times10^{-10}$. From the numerical illustration, the 120 harmonic terms can give an accurate analytical solution of period-4 motion in the vicinity of $\Omega = 38.56,$. That is, the accuracy of the analytical solution is about 10^{-11}. The tradition perturbation method with only few terms cannot achieve such an accurate solution.

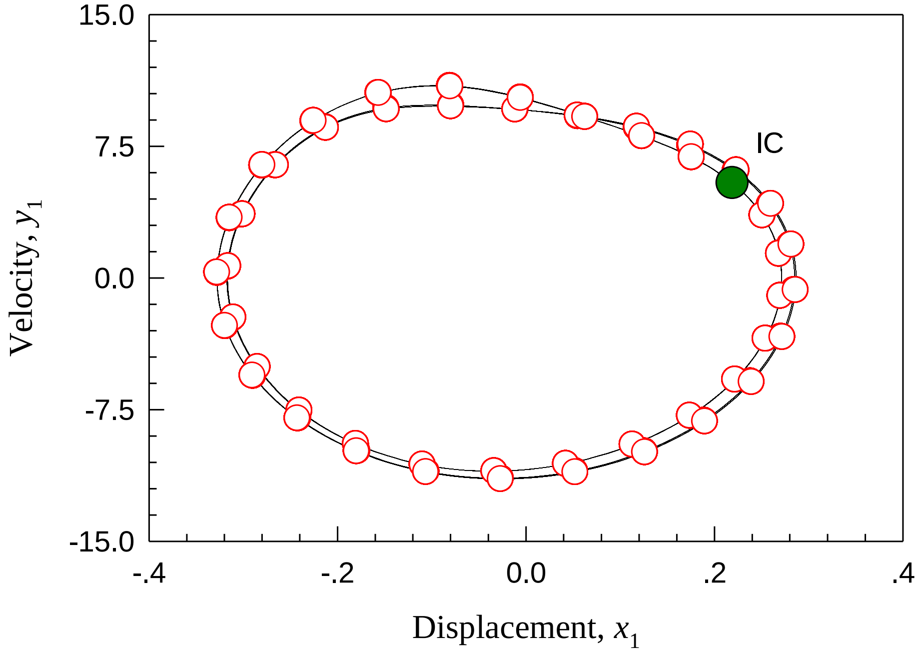


(i)

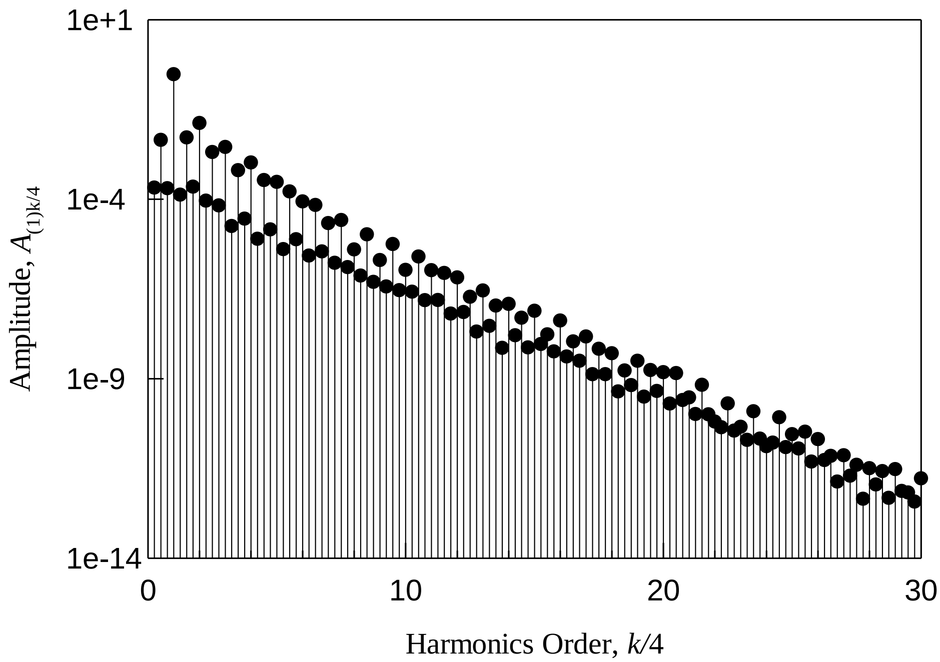


(ii)

Figure 5.5: Stable period-4 motion of nonlinear cable structure in transverse direction ($\Omega = 38.56$, HB120): (i) displacement x_1, (ii) velocity y_1; (iii) trajectory (x_1, y_1), (iv) harmonic amplitudes $A_{(1)k/4}$ ($k = 1, 2, \cdots, 120$). Motion in torsional direction: (v) displacement x_2, (vi) velocity y_2; (vii) trajectory (x_2, y_2), (viii) harmonic amplitudes $A_{(2)k/2}$ ($k = 1, 2, \cdots, 60$). Initial conditions $(x_{10}, y_{10}) = (.218801, 5.424894)$ and $(x_{20}, y_{20}) = (.253783, 36.973190)$.

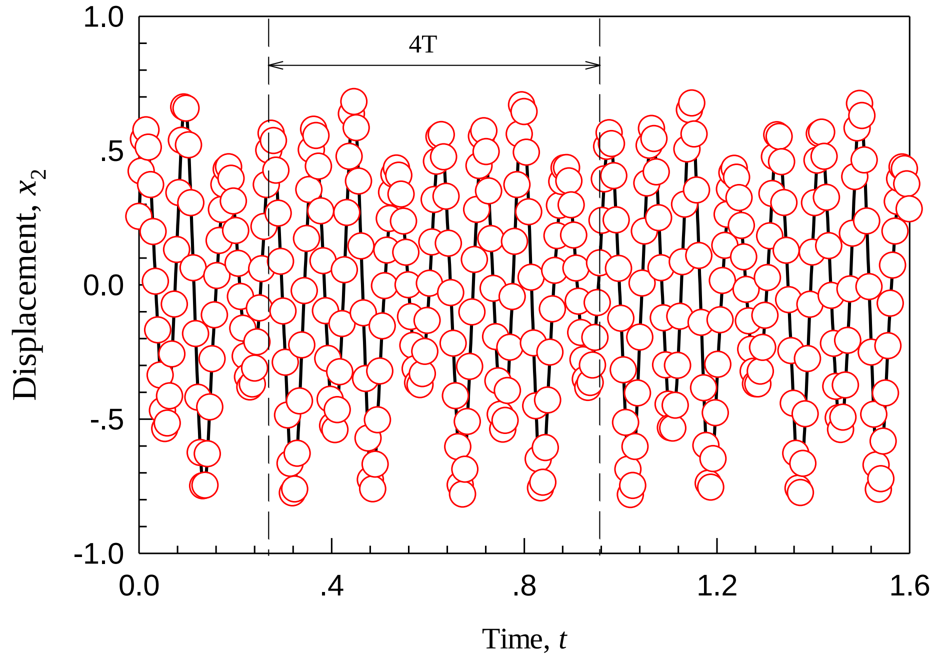


(iii)

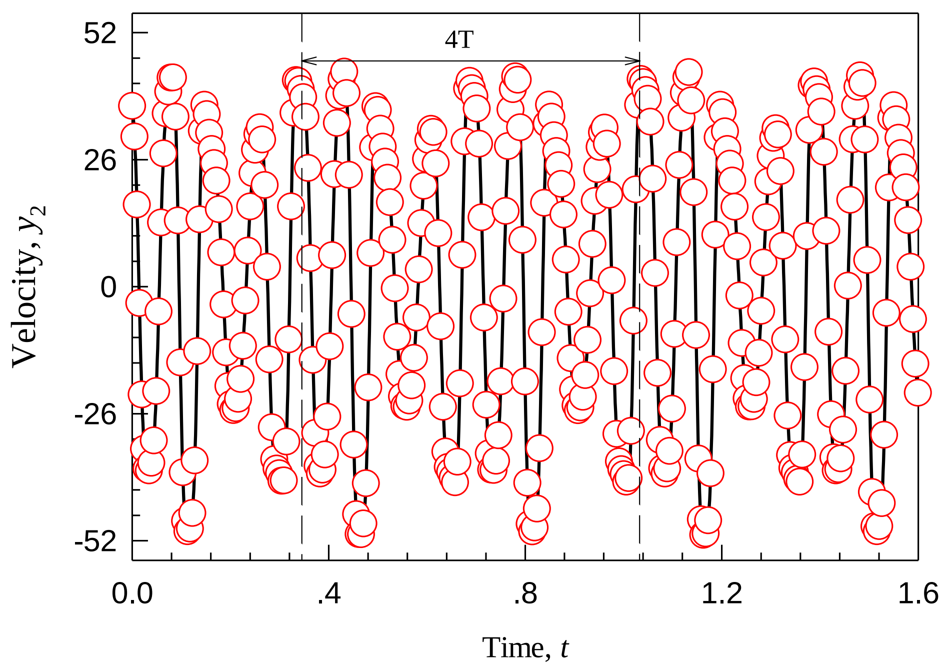


(iv)

Figure 5.5 Continued

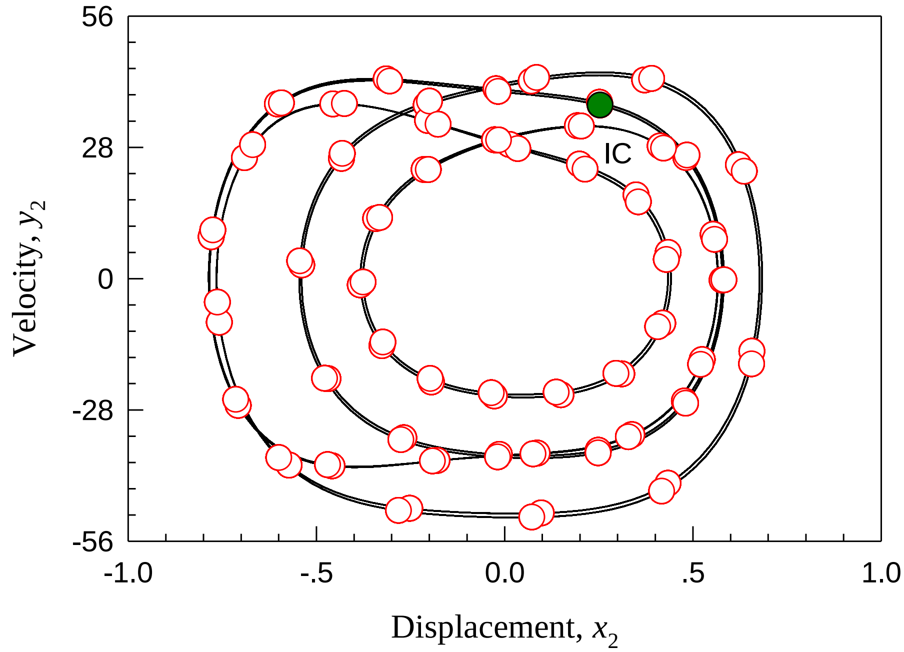


(v)

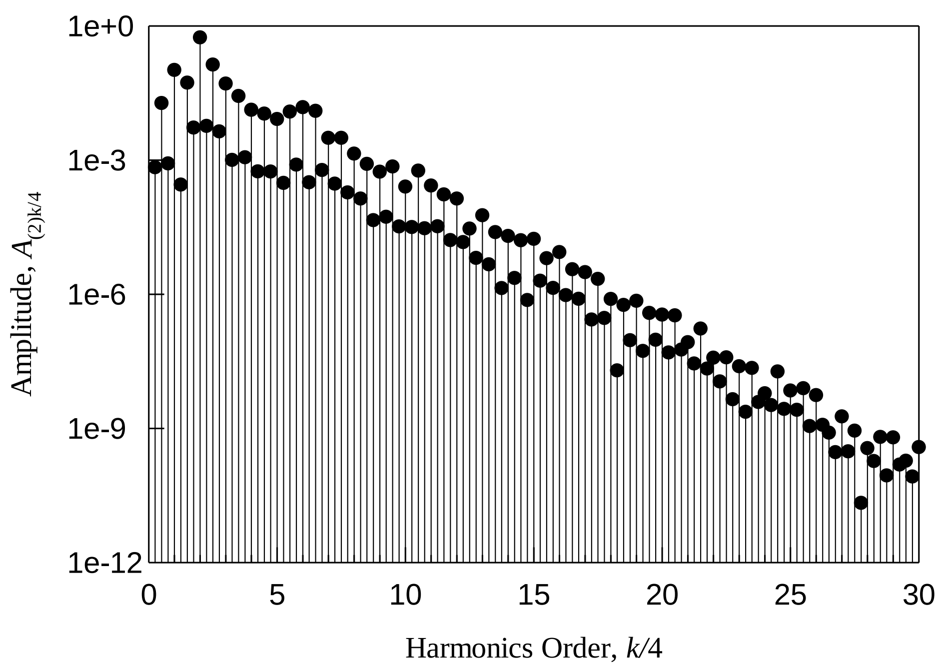


(vi)

Figure 5.5 Continued



(vii)



(viii)

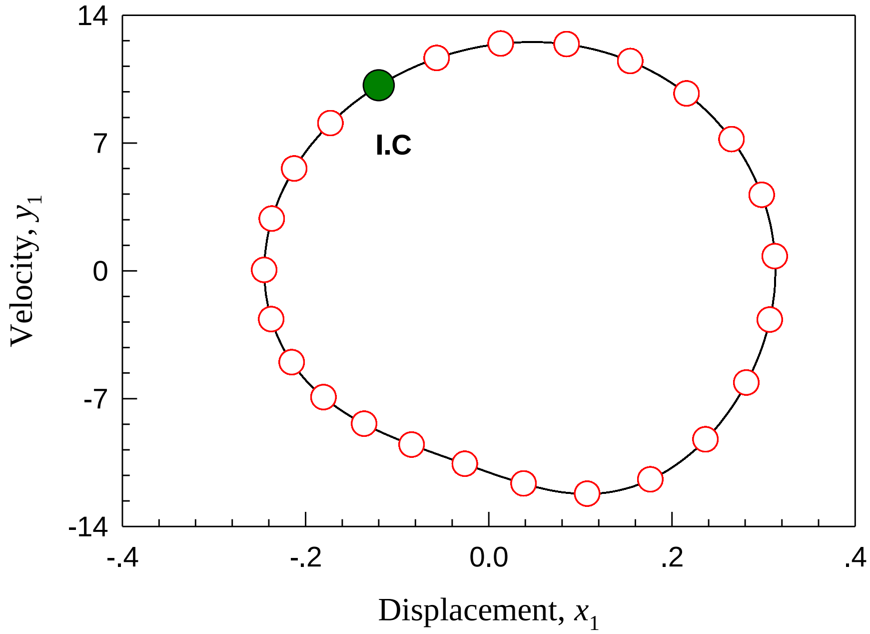
Figure 5.5 Continued

To illustrate periodic motions on other side of the same bifurcation trees of period-1 to chaos, the trajectories of period-1, perod-2 and period-4 motions are illustrated in Figure 5.6 (i)-(vi) for $\Omega = 43.56,\ 43.0,$ and $42.605,$ respectively. The initial conditions for numerical simulations of the three periodic motions are computed from the analytical solutions, as tabulated in Table 1. Since the excitation frequencies are quite close, the initial conditions for the three motions are also very close for the period-1, period-2 and period-4 motions.

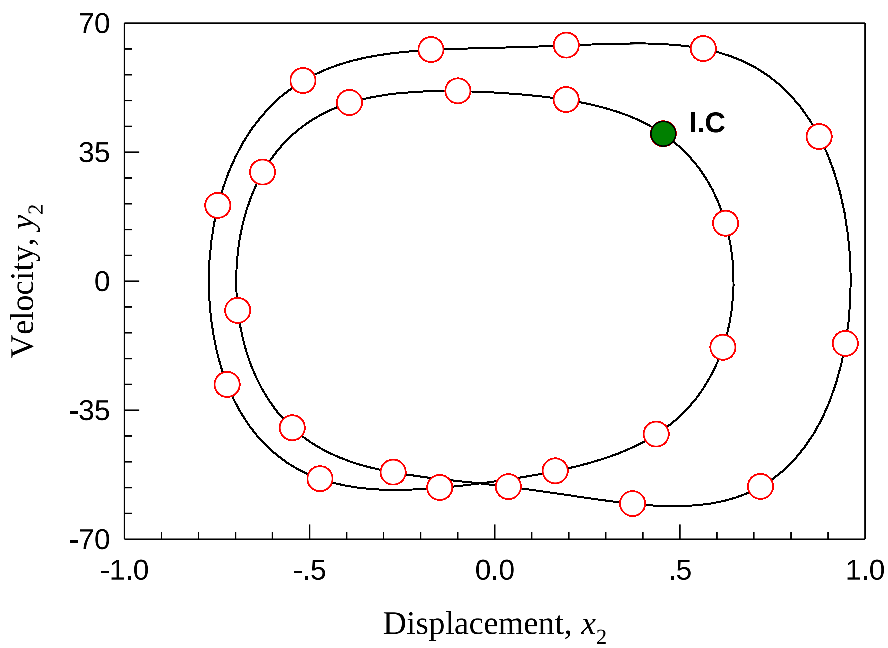
In Figure 5.6 (i) and (ii), the analytical solutions based on thirty harmonic terms (HB30) are determined for the period-1 motion, and the corresponding harmonic amplitudes are computed. The harmonic amplitudes decrease exponentially with increasing harmonic orders. The maximum and minimum harmonic amplitudes are $A_{(1)1} \approx .277506$ and $A_{(1)30} \approx 2.361\times 10^{-12}$ with $a_{10} = 0.0228$ for transverse motion, $A_{(2)1} \approx 0.1003$, and $A_{(2)30} \approx 4.0282\times 10^{-10}$ with $a_{20} \approx 0.0316$ for the torsional motion. The centers of the trajectories in transverse and torsional directions are at $x_1 \approx 0.0344$ and $x_2 \approx 0.0925$ that are not on the origin of the coordinate system. The trajectories of period-1 motions at $\Omega = 43.56$ in both directions are very similar to the period-1 motions at $\Omega = 35.4$. The magnitudes of the vibrations at $\Omega = 43.56$ are greater than at $\Omega = 35.4$. From the harmonic amplitudes, the analytical solutions of period-1 motions are very accurate. In Figure 5.6 (i) and (ii), the period-1 motion in the transverse direction has only cycle in phase plane, and the period-1 motion in the torsional motion has two cycles in phase plane. The numerical and analytical solutions match very well. The modal shape cannot be similar to what one thinks in the traditional perturbation analysis.

In Figure 5.6 (iii) and (iv), the analytical solutions based on sixty harmonic terms (HB60) are determined for the period-2 motion, and the corresponding harmonic amplitudes also are computed. The primary harmonic amplitudes $A_{(1)2l/2}$ and $A_{(2)2l/2}$ ($l=1,2,\cdots,30$) decrease exponentially with harmonic orders. The harmonic amplitudes $A_{(1)(2l-1)/2}$ and $A_{(2)(2l-1)/2}$ ($l=1,2,\cdots,30$) possesses a little wavy and exponential decrease with harmonic orders, which are effects on the period-2 motion derived from the period-1 motion. The maximum and minimum primary harmonic amplitudes are $A_{(1)2/2}\approx 0.2881$ and $A_{(1)60/2}\approx 3.1968\times 10^{-12}$ with $a_{10}^{(2)}=0.0223$ for the transverse motion, $A_{(2)4/2}\approx 0.7028$ and $A_{(2)60/2}\approx 5.9425\times 10^{-10}$ with $a_{20}^{(2)}\approx 0.0251$ for the torsional motion. The centers of the trajectories for the transverse and torsional motions are at $x_1=0.032$ and $x_2\approx 0.0637$. The maximum and minimum of harmonic amplitudes for the period-2 motion are $A_{(1)1/2}\approx 5.704\times 10^{-3}$ and $A_{(1)59/2}\approx 4..3326\times 10^{-12}$ for the transverse motion, $A_{(2)5/2}\approx 0.1504$ ($A_{(2)1/2}\approx 0.0149$) and $A_{(2)59/2}\approx 7.0516\times 10^{-10}$ for the torsional motion. From the harmonic amplitudes, the analytical solutions of period-2 motions are still very accurate. In Figure 5.6 (iii) and (iv), the period-2 motion in the transverse direction has two cycles in phase plane, and the period-2 motion in the torsional direction has four cycles in phase plane.

In Figure 5.6 (v) and (vi), the analytical solutions based on 120 harmonic terms (HB120) are determined for the period-4 motion, and the corresponding harmonic amplitudes are computed as well. The primary harmonic amplitudes $A_{(1)4l/4}$ and $A_{(2)4l/4}$ ($l=1,2,\cdots,30$) possesses a little wavy and exponential decrease with harmonic orders. The harmonic amplitudes $A_{(1)2(2l-1)/4}$ and $A_{(2)2(2l-1)/4}$ ($l=1,2,\cdots,30$) decrease wavily and exponentially with harmonic orders. The harmonic amplitudes $A_{(1)(4l-3)/4}$ with $A_{(1)(4l-1)/4}$, and $A_{(2)(4l-3)/4}$ with $A_{(2)(4l-1)/4}$ ($l=1,2,\cdots,30$) experience strongly wavy and exponential decreases with harmonic orders, which is for period-4 motion only. The maximum and minimum of primary harmonic amplitudes are $A_{(1)4/4} \approx 0.2957$ and $A_{(1)120/4} \approx 8.0966\times10^{-13}$ with $a_{10}^{(4)} = -0.0214$ for transverse motion, $A_{(2)4/4} \approx 0.1070$ and $A_{(2)120/4} \approx 1.5225\times10^{-11}$ with $a_{20}^{(4)} \approx -0.0246$ for the torsional motion. The centers of the trajectories for transverse and torsional motions are at $x_1 = -0.0281$ and $x_2 \approx -0.06875$ respectively. The maximum and minimum of second primary harmonic amplitudes are $A_{(1)2/4} \approx 8.0146\times10^{-3}$ ($A_{(1)6/4} \approx 6.1168\times10^{-3}$) and $A_{(1)118/4} \approx 5.0693\times10^{-12}$ for the transverse motion, $A_{(2)10/4} \approx 0.2111$ ($A_{(2)6/4} \approx 0.0938$) and $A_{(2)118/4} \approx 9.1607\times10^{-10}$ for the torsional motion. The maximum and minimum of harmonic amplitudes for period-4 motion only are $A_{(1)3/4} \approx 1.4834\times10^{-3}$ ($A_{(1)1/4} \approx 1.0022\times10^{-3}$) and $A_{(1)117/4} \approx 2.2607\times10^{-12}$ for the transverse motion, $A_{(2)7/4} \approx 0.231$ ($A_{(2)9/4} \approx 0.02$) and $A_{(2)117/4} \approx 3.7012\times10^{-10}$ for the torsional motion. From the harmonic amplitudes, the analytical solutions of period-4 motions are very accurate. In Figure 5.6 (v) and (vi), the period-4 motion in the transverse direction has four cycles in phase plane, and the period-2 motion in the torsional direction has eight cycles in phase plane. However, the quantity levels of harmonic amplitudes $A_{(1)(4l-3)/4}$, $A_{(1)(4l-1)/4}$, $A_{(2)(4l-3)/4}$, and $A_{(2)(4l-1)/4}$ ($l = 1,2,\cdots,30$) are small compared to the harmonic amplitudes of $A_{(1)2(2l-1)/4}$ and $A_{(2)2(2l-1)/4}$ ($l = 1,2,\cdots,30$), thus the period-4 motion is very close to the period-2 motion.

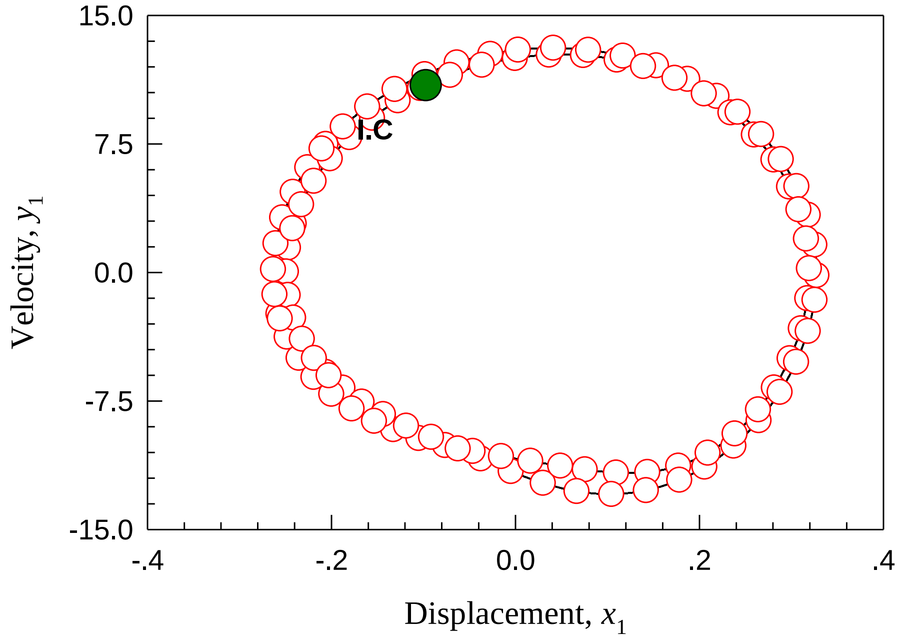


(i)

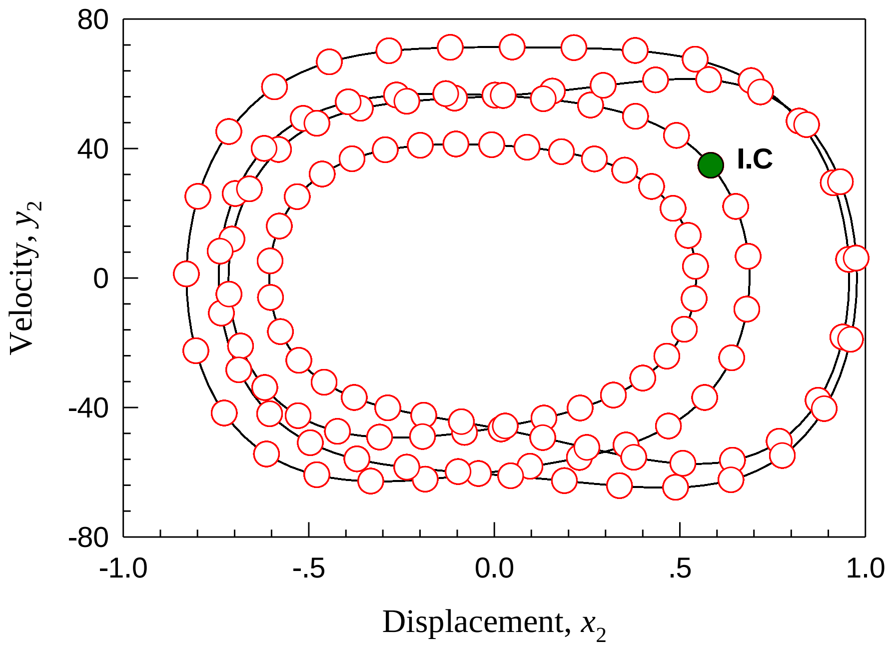


(ii)

Figure 5.6: Stable period-1 to perid-4 motion of nonlinear cable structure in both transverse and torsional directions on the other side of the bifurcation tree. Stable period-1 motion ($\Omega = 43.56$, HB30): (i) Trajectory in transverse direction (x_1, y_1), (ii) Trajectory in torsional direction (x_2, y_2) Stable period-2 motion ($\Omega = 43.0$, HB=60): (iii) Trajectory in transverse direction (x_1, y_1), (iv) Trajectory in torsional direction (x_2, y_2). Stable period-4 motion ($\Omega = 42.605$, HB=120): (v) Trajectory in transverse direction (x_1, y_1), (vi) Trajectory in torsional direction (x_2, y_2).

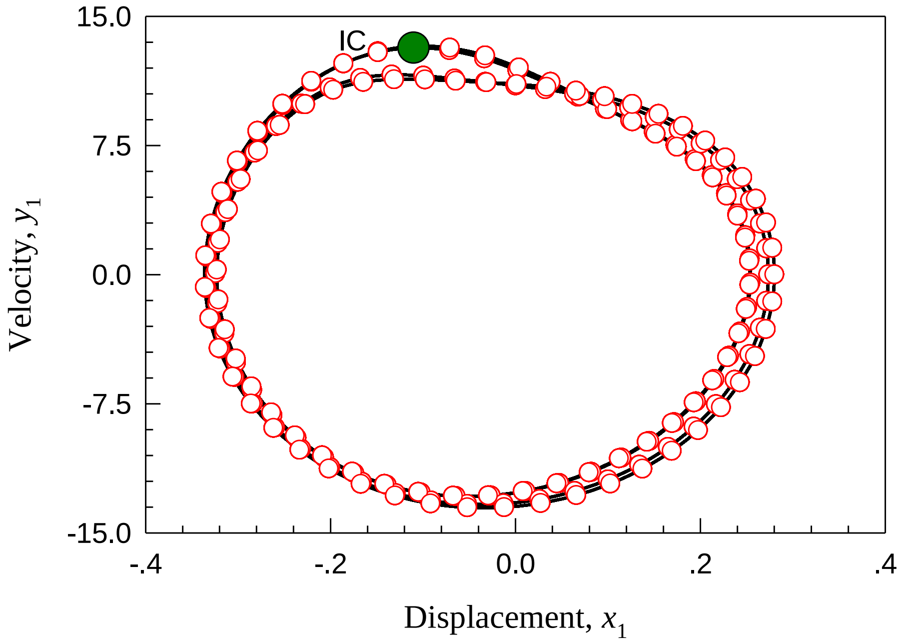


(iii)

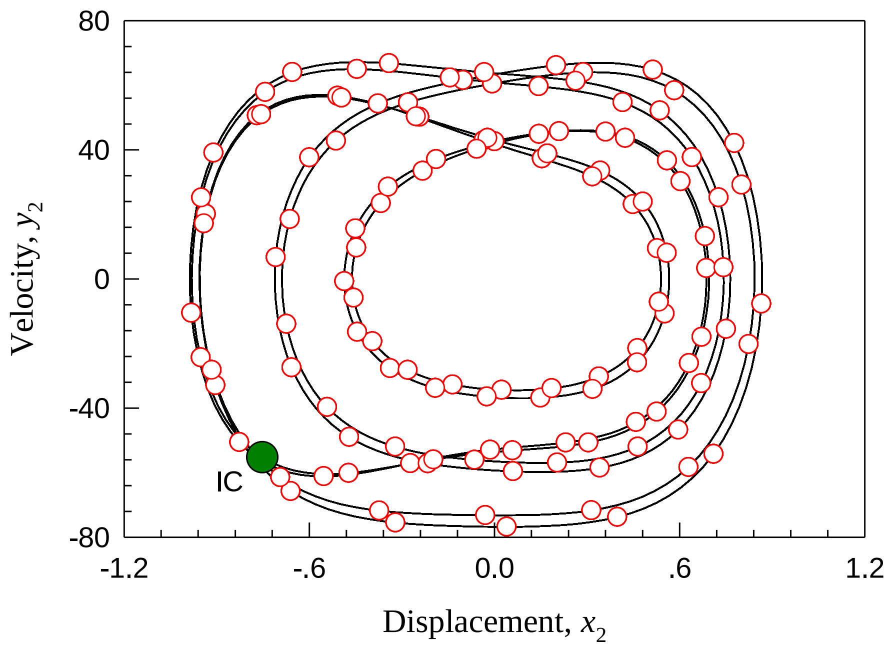


(iv)

Figure 5.6 Continued



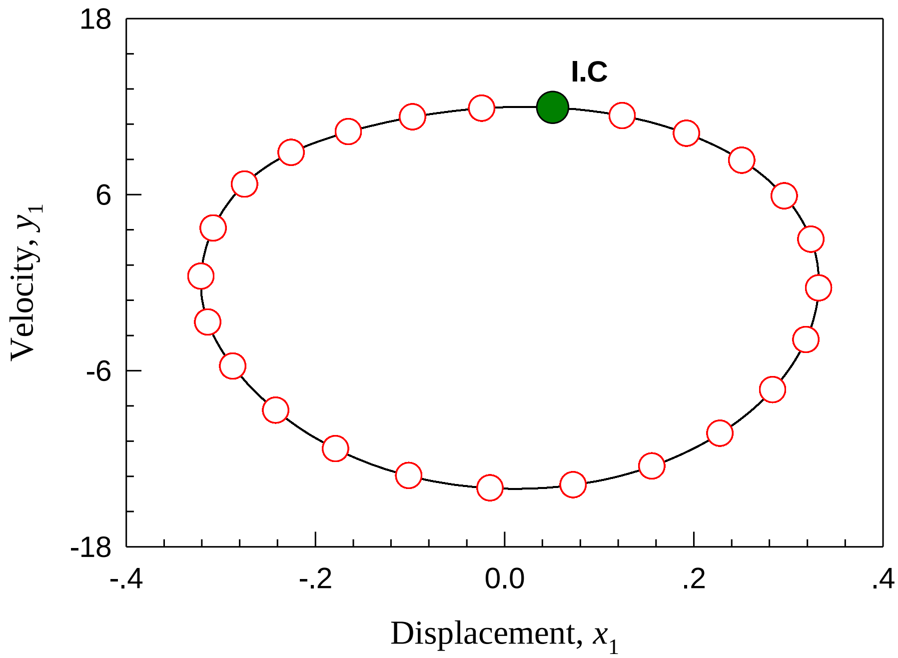
(v)



(vi)

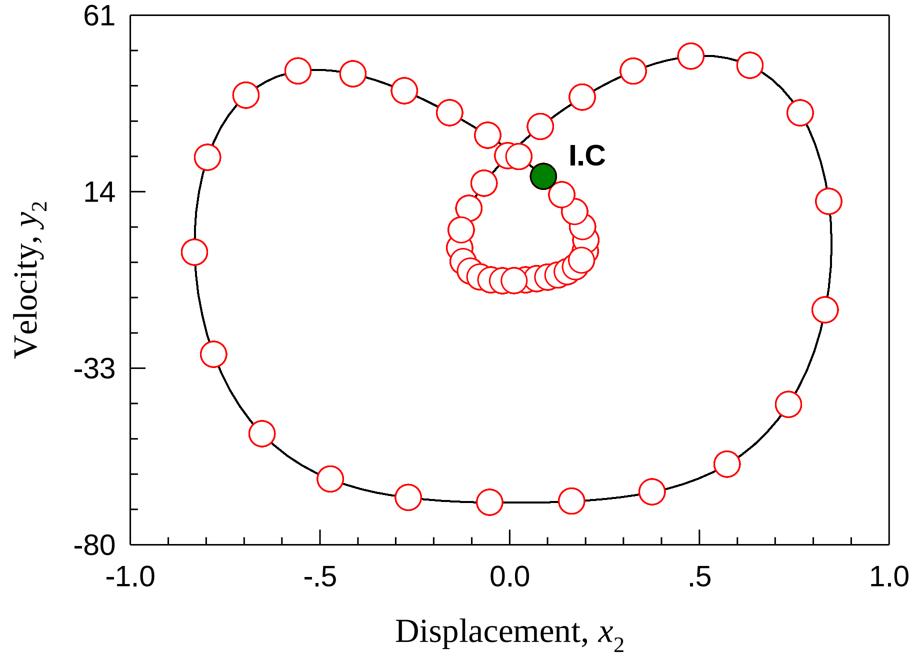
Figure 5.6 Continued

On the branch of pure period-1 motion, two period-1 motion at $\Omega = 37.55$ and $\Omega = 40.0$ are presented in Figure 5.7. Both periodic motions are asymmetric period-1 motions. Since they are from different branch, so pattern of trajectories are different from the period-1 motions at $\Omega = 35.40$ and $\Omega = 43.56$. The precision of analytical solutions for period-1 motion in the transverse direction is 10^{-12} and 10^{-10} for torsional motion. Therefore the analytical solutions of cable vibrations are very accurate.

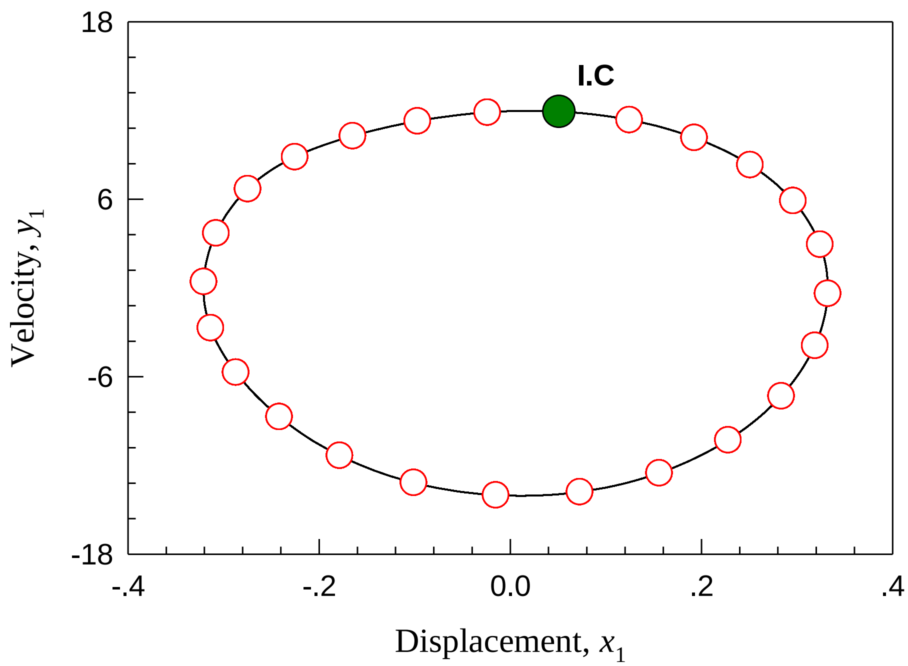


(i)

Figure 5.7: Stable symmetric period-1 motion of nonlinear cable structure ($\Omega = 37.55$, HB30): (i) trajectory (x_1, y_1) in transverse direction, (ii) trajectory (x_2, y_2) in torsional direction. Stable symmetric period-1 motion of nonlinear cable structure ($\Omega = 40.0$, HB30): (iii) trajectory (x_1, y_1) in transverse direction, (iv) trajectory (x_2, y_2) in torsional direction.

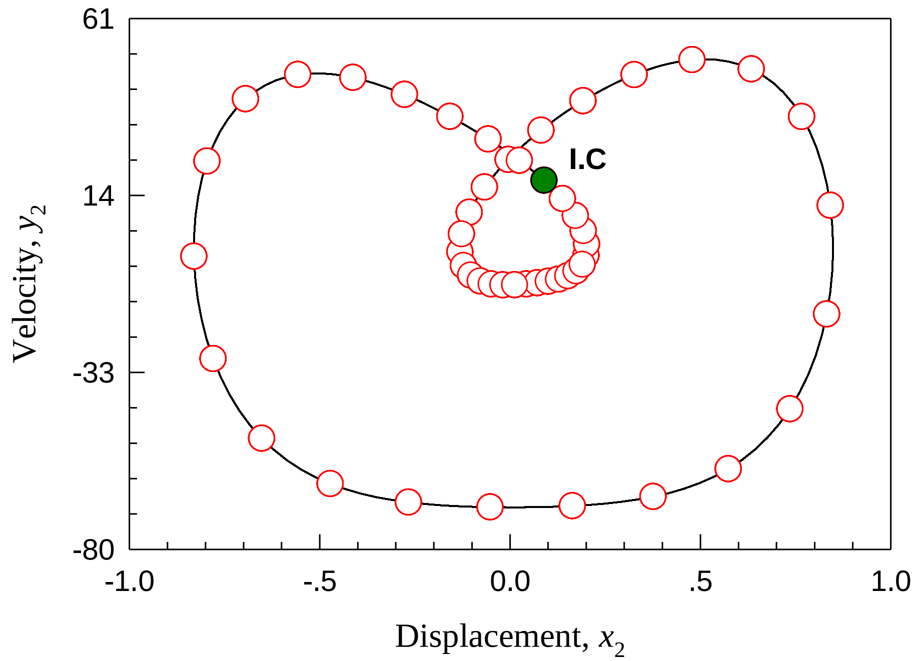


(ii)



(iii)

Figure 5.7 Continued



(iv)

Figure 5.7 Continued

Table 5.1 Input data for numerical simulations

Fig 5.6	Ω	x_{10}	y_{10}	x_{20}	y_{20}	Type	Stability
(i),(ii)	43.56	-0.119849	10.148384	0.455719	39.892679	P-1 (HB30)	Stable
(iii),(iv)	43.0	-0.097281	10.916667	0.583876	34.731342	P-2 (HB60)	Stable
(v),(vi)	42.605	-0.110236	13.168890	-0.751864	-55.28607	P-4 (HB120)	Stable

Table 5.2 Input data for numerical simulations

Fig 5.7	Ω	x_{10}	y_{10}	x_{20}	y_{20}	Type	Stability
(i),(ii)	37.55	0.184514	9.413522	0.153034	2.262215	P-1 (HB30)	Stable
(iii),(iv)	40.0	0.050939	11.930684	0.089221	17.992086	P-1 (HB30)	Stable

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Summary

A two-degree-of-freedom model of galloping oscillation has been developed for a single iced transmission cable which might be vibrate transversely and torsionally. The method of generalized harmonic balance method is systematically introduced and applied to investigate the periodic motions and limit cycles.

The periodic motions of a linear cable structure is investigated analytically by using the generalized harmonic balance method. The analytical bifurcation trees of periodic motions are obtained for square prism section. The stability analysis is also performed for the periodic motions. Both stable and unstable analytical solutions of periodic motions are presented. The limit cycle, if galloping vibrations exist, are obtained. Finally, the phase trajectories, displacement, and velocity time history plots for different periodic motions are illustrated.

The periodic motions of a nonlinear cable structure is carried out by using the generalized harmonic balance method. The analytical routes from period-1 motions to chaos are obtained for square prism section. The stabilities are analyzed. Stable period-1 motions to period-4 motions are demonstrated in the phase space and time domain. Such a nonlinear phenomenon will be helpful to understand mechanism of galloping vibrations.

6.2 Future work

The following working are suggested.

1. An investigation of the periodic motions of elastic structures of other cross sections can be completed immediately. It can be used to evaluate the effects of different cross sections on periodic motions and galloping vibrations.

2. The analytical solutions of periodic motions in high DOF models can be investigated by using the generalized harmonic balance method. However, it needs a large amount of computation work. Other techniques can be used instead of the method of generalized harmonic balance.

3. A bundle of conductors rather than a single conductor can be investigated and studied in the future.

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APPENDICES

APPENDIX A

A.1 Mathematical expression for each of the term after averaging for the constant term.

$$
\begin{aligned}
f_1^1 = {} & \dot{a}_{20}^3 + \frac{3}{2}\dot{a}_{20}\sum_{i=1}^{N}(B_{i/m}^2 + C_{i/m}^2) + \frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{i/m}C_{j/m}B_{k/m}(\delta_{i+k}^{j} + \delta_{k+j}^{i} - \delta_{i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \\
& B_{i/m}B_{j/m}B_{k/m}(\delta_{i+k}^{j} + \delta_{k+j}^{i} + \delta_{i+j}^{k})
\end{aligned}
$$

$$
\begin{aligned}
f_1^2 = {} & \dot{a}_{20}^2\dot{a}_{10} + \dot{a}_{20}\sum_{i=1}^{N}(B_{i/m}P_{i/m} + C_{i/m}Q_{i/m}) + \frac{1}{2}\dot{a}_{10}\sum_{i=1}^{N}(B_{i/m}^2 + C_{i/m}^2) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} B_{i/m}B_{j/m}P_{k/m}(\delta_{i+k}^{j} \\
& + \delta_{k+j}^{i} + \delta_{i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{i/m}C_{j/m}P_{k/m}(\delta_{i+k}^{j} + \delta_{k+j}^{i} - \delta_{i+j}^{k}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} B_{i/m}C_{j/m}Q_{k/m}(\delta_{i+k}^{j} \\
& + \delta_{i+j}^{k} - \delta_{i+j}^{i})
\end{aligned}
$$

$$
\begin{aligned}
f_1^3 = {} & a_{20}\dot{a}_{20}^2 + \frac{1}{2}a_{20}\sum_{i=1}^{N}(B_{i/m}^2 + C_{i/m}^2) + \dot{a}_{20}\sum_{i=1}^{N}(b_{2i/m}B_{i/m} + c_{2i/m}C_{i/m}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}B_{j/m}B_{k/m}(\delta_{i+k}^{j} \\
& + \delta_{k+j}^{i} + \delta_{i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}C_{j/m}C_{k/m}(\delta_{i+k}^{j} + \delta_{i+j}^{k} - \delta_{i+j}^{i}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}B_{j/m}C_{k/m}(\delta_{j+k}^{i} \\
& + \delta_{i+j}^{k} - \delta_{i+k}^{j})
\end{aligned}
$$

$$
\begin{aligned}
f_1^4 = {} & \dot{a}_{20}\dot{a}_{10}^2 + \frac{1}{2}\dot{a}_{20}\sum_{i=1}^{N}(P_{i/m}^2 + Q_{i/m}^2) + \dot{a}_{10}\sum_{i=1}^{N}(B_{i/m}P_{i/m} + C_{i/m}Q_{i/m}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} B_{i/m}P_{j/m}P_{k/m}(\delta_{j+k}^{i} \\
& + \delta_{i+j}^{k} + \delta_{i+k}^{j}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} B_{i/m}Q_{j/m}Q_{k/m}(\delta_{i+j}^{k} + \delta_{i+k}^{j} - \delta_{j+k}^{i}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{i/m}P_{j/m}Q_{k/m}(\delta_{i+j}^{k} \\
& + \delta_{j+k}^{i} - \delta_{i+k}^{j})
\end{aligned}
$$

$$
\begin{aligned}
f_1^5 = {} & a_{20}\dot{a}_{20}\dot{a}_{10} + \frac{1}{2}a_{20}\sum_{i=1}^{N}(B_{i/m}P_{i/m} + C_{i/m}Q_{i/m}) + \frac{1}{2}\dot{a}_{10}\sum_{i=1}^{N}(b_{2i/m}B_{i/m} + c_{2i/m}C_{i/m}) + \frac{1}{2}\dot{a}_{20}\sum_{i=1}^{N}(b_{2i/m}P_{i/m} \\
& + c_{2i/m}Q_{i/m}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}B_{j/m}P_{k/m}(\delta_{j+k}^{i} + \delta_{i+j}^{k} + \delta_{i+k}^{j}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}C_{j/m}Q_{k/m}(\delta_{i+j}^{k} \\
& + \delta_{i+k}^{j} - \delta_{j+k}^{i}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}B_{j/m}Q_{k/m}(\delta_{i+j}^{k} - \delta_{i+k}^{j} + \delta_{j+k}^{i}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}C_{j/m}P_{k/m}(\delta_{j+k}^{i} \\
& - \delta_{i+j}^{k} + \delta_{i+k}^{j})
\end{aligned}
$$

$$
\begin{aligned}
f_1^6 = {} & a_{20}^2\dot{a}_{20} + a_{20}\sum_{i=1}^{N}(b_{2i/m}B_{i/m} + c_{2i/m}C_{i/m}) + \frac{1}{2}\dot{a}_{20}\sum_{i=1}^{N}(b_{2i/m}^2 + c_{2i/m}^2) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}b_{2j/m}B_{k/m}(\delta_{j+k}^{i} \\
& + \delta_{i+j}^{k} + \delta_{i+k}^{j}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}c_{2j/m}C_{k/m}(\delta_{i+j}^{k} + \delta_{i+k}^{j} - \delta_{j+k}^{i}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}c_{2j/m}B_{k/m}(\delta_{i+k}^{j} \\
& + \delta_{j+k}^{i} - \delta_{i+j}^{k})
\end{aligned}
$$

$$f_1^8 = \dot{a}_{10}^3 + \frac{3}{2}\dot{a}_{10}\sum_{i=1}^{N}(P_{i/m}^2 + Q_{i/m}^2) + \frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}Q_{i/m}Q_{j/m}P_{k/m}(\delta_{j+k}^i - \delta_{i+j}^k + \delta_{i+k}^j) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}P_{i/m}P_{j/m}$$

$$P_{k/m}(\delta_{i+j}^k + \delta_{i+k}^j + \delta_{j+k}^i)$$

$$f_1^9 = \dot{a}_{10}^2 a_{20} + \frac{1}{2}a_{20}\sum_{i=1}^{N}(P_{i/m}^2 + Q_{i/m}^2) + \dot{a}_{10}\sum_{i=1}^{N}(b_{2i/m}P_{i/m} + c_{2i/m}Q_{i/m}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}P_{j/m}P_{k/m}(\delta_{j+k}^i$$

$$+\delta_{k+j}^k + \delta_{i+k}^j) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}Q_{j/m}Q_{k/m}(\delta_{k+j}^k + \delta_{i+k}^j - \delta_{j+k}^i) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}P_{j/m}Q_{k/m}(\delta_{i+j}^k$$

$$-\delta_{i+k}^j + \delta_{j+k}^i)$$

$$f_1^{10} = a_{20}^2\dot{a}_{10} + a_{20}\sum_{i=1}^{N}(b_{2i/m}P_{i/m} + c_{2i/m}Q_{i/m}) + \frac{1}{2}\dot{a}_{10}\sum_{i=1}^{N}(b_{2i/m}^2 + c_{2i/m}^2) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}c_{2j/m}Q_{k/m}$$

$$(\delta_{k+j}^k + \delta_{i+k}^j - \delta_{j+k}^i) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}b_{2j/m}P_{k/m}(\delta_{k+j}^k + \delta_{i+k}^j + \delta_{j+k}^i) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}c_{2j/m}$$

$$P_{k/m}(\delta_{i+k}^j + \delta_{j+k}^i - \delta_{i+j}^k)$$

$$f_1^{12} = a_{20}^3 + \frac{3}{2}a_{20}\sum_{i=1}^{N}(b_{2i/m}^2 + c_{2i/m}^2) + \frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}c_{2j/m}b_{2k/m}(\delta_{j+k}^i - \delta_{i+j}^k + \delta_{i+k}^j) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}$$

$$b_{2i/m}b_{2j/m}b_{2k/m}(\delta_{i+j}^k + \delta_{i+k}^j + \delta_{j+k}^i)$$

$$f_1^7 = \dot{a}_{20},\ f_1^{11} = \dot{a}_{10},\ f_1^{13} = a_{20},\ f_{11}^{14} = \dot{a}_{10},\ f_{11}^{15} = a_{10},\ f_{12}^{14} = \dot{a}_{20},\ f_{12}^{15} = a_{20}$$

where,

$$B_{i/m} = \dot{b}_{2i/m} + \frac{i\Omega c_{2i/m}}{m},\ C_{i/m} = \dot{c}_{2i/m} - \frac{i\Omega b_{2i/m}}{m},\ P_{i/m} = \dot{b}_{1i/m} + \frac{i\Omega c_{1i/m}}{m},\ Q_{i/m} = \dot{c}_{1i/m} - \frac{i\Omega b_{1i/m}}{m}$$

A.2 Mathematical expression for each of the term after averaging for the cosine term.

$$f_2^1 = 3\dot{a}_{20}^2 B_{n/m} + \frac{3}{2}\dot{a}_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}B_{i/m}B_{j/m}(\delta_{i+n}^j + \delta_{n+j}^i + \delta_{i+j}^n) + \frac{3}{2}\dot{a}_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}C_{i/m}C_{j/m}(\delta_{i+n}^j + \delta_{n+j}^i - \delta_{i+j}^n) +$$

$$\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}C_{i/m}C_{j/m}B_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^j - \delta_{i+j+k}^n + \delta_{n+j+k}^i + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} - \delta_{n+i+j}^k) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}$$

$$B_{i/m}B_{j/m}B_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^j + \delta_{i+j+k}^n + \delta_{n+j+k}^i + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} + \delta_{n+i+j}^k)$$

$$f_2^2 = \dot{a}_{20}^2 P_{n/m} + 2\dot{a}_{20}\dot{a}_{10}B_{n/m} + \dot{a}_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}B_{j/m}P_{k/m}(\delta_{k+n}^j + \delta_{j+n}^k + \delta_{k+j}^n) + \dot{a}_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}C_{j/m}Q_{k/m}(\delta_{k+n}^j + \delta_{n+j}^k$$

$$-\delta_{k+j}^n) + \frac{\dot{a}_{10}}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}B_{i/m}B_{j/m}(\delta_{j+n}^i + \delta_{n+i}^j + \delta_{i+j}^n) + \frac{\dot{a}_{10}}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}C_{i/m}C_{j/m}(\delta_{j+n}^i + \delta_{n+i}^j - \delta_{i+j}^n) + \frac{1}{4}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}C_{i/m}C_{j/m}P_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^j - \delta_{i+j+k}^n + \delta_{n+j+k}^i + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} - \delta_{n+i+j}^k) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}$$

$$B_{i/m}C_{j/m}Q_{k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}-\delta_{n+j+k}^{i}-\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})$$

$$f_2^3=2\dot{a}_{20}a_{20}B_{n/m}+\dot{a}_{20}^2b_{2n/m}+\frac{1}{2}a_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}B_{j/m}B_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}+\delta_{k+j}^{n})+\frac{1}{2}a_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}C_{j/m}C_{k/m}(\delta_{k+n}^{j}$$

$$+\delta_{n+j}^{k}-\delta_{k+j}^{n})+\dot{a}_{20}\sum_{i=1}^{N}\sum_{k=1}^{N}b_{2i/m}B_{k/m}(\delta_{k+n}^{i}+\delta_{n+i}^{k}+\delta_{k+i}^{n})+\dot{a}_{20}\sum_{i=1}^{N}\sum_{k=1}^{N}c_{2i/m}C_{k/m}(\delta_{k+n}^{i}+\delta_{n+i}^{k}-\delta_{k+i}^{n})+$$

$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}B_{j/m}B_{k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}+\delta_{i+j+k}^{n}+\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}$$

$$b_{2i/m}C_{j/m}C_{k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}-\delta_{n+j+k}^{i}-\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}B_{j/m}C_{k/m}$$

$$(\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k}-\delta_{i+k}^{j+n}-\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n})$$

$$f_2^4=2\dot{a}_{20}\dot{a}_{10}P_{n/m}+\dot{a}_{10}^2B_{n/m}+\frac{1}{2}\dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}P_{j/m}P_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}+\delta_{k+j}^{n})+\frac{1}{2}\dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}Q_{j/m}Q_{k/m}(\delta_{k+n}^{j}$$

$$+\delta_{n+j}^{k}-\delta_{k+j}^{n})+\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N}B_{j/m}P_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}+\delta_{k+j}^{n})+\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N}C_{j/m}Q_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}-\delta_{k+j}^{n})+$$

$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}B_{i/m}P_{j/m}P_{k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}+\delta_{i+j+k}^{n}+\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}$$

$$B_{i/m}Q_{j/m}Q_{k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}-\delta_{n+j+k}^{i}-\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}C_{i/m}P_{j/m}Q_{k/m}$$

$$(\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k}-\delta_{i+k}^{j+n}-\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n})$$

$$f_2^5=a_{20}\dot{a}_{20}P_{n/m}+a_{20}\dot{a}_{10}B_{n/m}+\frac{1}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}B_{j/m}P_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}+\delta_{k+j}^{n})+\frac{1}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}C_{j/m}Q_{k/m}(\delta_{k+n}^{j}$$

$$+\delta_{n+j}^{k}-\delta_{k+j}^{n})+\dot{a}_{20}\dot{a}_{10}b_{2n/m}+\frac{1}{2}\dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2j/m}P_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}+\delta_{k+j}^{n})+\frac{1}{2}\dot{a}_{10}$$

$$\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2j/m}B_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}+\delta_{k+j}^{n})+\frac{1}{2}\dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2j/m}Q_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}-\delta_{k+j}^{n})+\frac{1}{2}\dot{a}_{10}$$

$$\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2j/m}C_{k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}-\delta_{k+j}^{n})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}B_{j/m}P_{k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}+\delta_{i+j+k}^{n}+\delta_{n+j+k}^{i}+$$

$$\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}C_{j/m}Q_{k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}-\delta_{n+j+k}^{i}-\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}$$

$$+\delta_{n+i+j}^{k})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}B_{j/m}Q_{k/m}(\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k}-\delta_{i+k}^{j+n}-\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n})$$

$$+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}C_{j/m}P_{k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}+\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}-\delta_{i+j}^{k+n}-\delta_{n+i+j}^{k})$$

$$f_2^6 = a_{20}^2 B_{n/m} + 2a_{20}\dot{a}_{20} b_{2n/m} + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m} B_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \frac{1}{2}\dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m} b_{2k/m}(\delta_{k+n}^{j}$$

$$+ \delta_{n+j}^{k} + \delta_{k+j}^{n}) + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2j/m} Q_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} - \delta_{k+j}^{n}) + \frac{1}{2}\dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2j/m} c_{2k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} - \delta_{k+j}^{n})$$

$$+ \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m} b_{2j/m} B_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) +$$

$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m} c_{2j/m} C_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} - \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}$$

$$\sum_{k=1}^{N} c_{2i/m} c_{2j/m} B_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k})$$

$$f_2^7 = \dot{b}_{2n/m} + \frac{n\Omega}{m} c_{2n/m}$$

$$f_2^8 = 3\dot{a}_{10}^2 P_{n/m} + \frac{3}{2}\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} P_{j/m} P_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \frac{3}{2}\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} Q_{j/m} Q_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} - \delta_{k+j}^{n}) +$$

$$\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} Q_{i/m} Q_{j/m} P_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k}) +$$

$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} P_{i/m} P_{j/m} P_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k})$$

$$f_2^9 = 2\dot{a}_{10}\dot{a}_{20} P_{n/m} + \frac{1}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} P_{j/m} P_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \frac{1}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} Q_{j/m} Q_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} - \delta_{k+j}^{n})$$

$$+ \dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m} P_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2j/m} Q_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} - \delta_{k+j}^{n}) + \dot{a}_{10}^2 b_{2n/m} +$$

$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m} P_{j/m} P_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) +$$

$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m} Q_{j/m} Q_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} - \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) +$$

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m} P_{j/m} Q_{k/m}(\delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k} - \delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n})$$

$$f_2^{10} = a_{20}^2 P_{n/m} + 2\dot{a}_{10} a_{20} b_{2n/m} + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m} P_{k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} + \delta_{k+j}^{n}) + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2j/m} Q_{k/m}$$

$$(\delta_{k+n}^{j} + \delta_{n+j}^{k} - \delta_{k+j}^{n}) + \frac{1}{2}\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m} b_{2k/m}(\delta_{k+n}^{j} + \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \frac{1}{2}\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2j/m} c_{2k/m}$$

$$(\delta_{k+n}^{j} + \delta_{n+j}^{k} - \delta_{k+j}^{n}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m} c_{2j/m} Q_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} - \delta_{n+i}^{k+j}$$

$$+ \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m} b_{2j/m} P_{k/m}(\delta_{i+k}^{j+n} + \delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n}$$

$$+\delta^{k}_{n+i+j})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}c_{2j/m}P_{k/m}(\delta^{j+n}_{i+k}+\delta^{j}_{i+k+n}-\delta^{n}_{i+j+k}+\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}-\delta^{k+n}_{i+j}-\delta^{k}_{n+i+j})$$

$$f_2^{12}=3a_{20}^2b_{2n/m}+\frac{3}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2j/m}b_{2k/m}(\delta^{j}_{k+n}+\delta^{k}_{n+j}+\delta^{n}_{k+j})+\frac{3}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2j/m}c_{2k/m}(\delta^{j}_{k+n}+\delta^{k}_{n+j}$$

$$-\delta^{n}_{k+j})+\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}c_{2j/m}b_{2k/m}(\delta^{j+n}_{i+k}+\delta^{j}_{i+k+n}-\delta^{n}_{i+j+k}+\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}-\delta^{k+n}_{i+j}-\delta^{k}_{n+i+j})+$$

$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}b_{2j/m}b_{2k/m}(\delta^{j+n}_{i+k}+\delta^{j}_{i+k+n}+\delta^{n}_{i+j+k}+\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}+\delta^{k+n}_{i+j}+\delta^{k}_{n+i+j})$$

$$f_2^{11}=\dot{b}_{1n/m}+\frac{n\Omega}{m}c_{1n/m},\ f_2^{13}=b_{2n/m},\ f_{21}^{14}=\dot{b}_{1n/m}+\frac{n\Omega}{m}c_{1n/m},\ f_{21}^{15}=b_{1n/m},\ f_1^{16}=\delta^{m}_{n},$$

$$f_{22}^{14}=\dot{b}_{2n/m}+\frac{n\Omega}{m}c_{2n/m},\ f_{22}^{15}=b_{2n/m}$$

A.3 Mathematical expression for each of the term after averaging for the sine term.

$$f_3^1=3\dot{a}_{20}^2C_{n/m}+3\dot{a}_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}B_{i/m}C_{j/m}(\delta^{j}_{i+n}+\delta^{n}_{i+j}-\delta^{i}_{n+j})+\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}B_{i/m}C_{j/m}B_{k/m}(\delta^{j}_{i+k+n}+\delta^{n}_{i+j+k}$$

$$-\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}+\delta^{k+n}_{i+j}-\delta^{k}_{n+i+j}-\delta^{j+n}_{i+k})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}C_{i/m}C_{j/m}C_{k/m}(\delta^{j+n}_{i+k}-\delta^{j}_{i+k+n}-\delta^{n}_{i+j+k}-\delta^{i}_{n+j+k}$$

$$+\delta^{k+j}_{n+i}+\delta^{k+n}_{i+j}-\delta^{k}_{n+i+j})$$

$$f_3^2=\dot{a}_{20}^2Q_{n/m}+2\dot{a}_{20}\dot{a}_{10}C_{n/m}+\dot{a}_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}B_{j/m}Q_{k/m}(\delta^{k}_{j+n}+\delta^{n}_{k+j}-\delta^{j}_{k+n})++\dot{a}_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}C_jP_k(\delta^{j}_{k+n}-\delta^{k}_{n+j}$$

$$+\delta^{n}_{k+j})+\dot{a}_{10}\sum_{i=1}^{N}\sum_{j=1}^{N}B_{i/m}C_{j/m}(\delta^{j}_{n+i}+\delta^{n}_{i+j}-\delta^{i}_{j+n})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}B_{i/m}B_{j/m}Q_{k/m}(\delta^{j+n}_{i+k}-\delta^{j}_{i+k+n}+\delta^{n}_{i+j+k}$$

$$-\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}-\delta^{k+n}_{i+j}+\delta^{k}_{n+i+j})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}C_{i/m}C_{j/m}Q_{k/m}(\delta^{j+n}_{i+k}-\delta^{j}_{i+k+n}-\delta^{n}_{i+j+k}-\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}$$

$$+\delta^{k+n}_{i+j}-\delta^{k}_{n+i+j})+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}B_{i/m}C_{j/m}P_{k/m}(\delta^{j}_{i+k+n}+\delta^{n}_{i+j+k}-\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}+\delta^{k+n}_{i+j}-\delta^{k}_{n+i+j}-\delta^{j+n}_{i+k})$$

$$f_3^3=2\dot{a}_{20}a_{20}C_{n/m}+\dot{a}_{20}^2c_{2n/m}+a_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}B_{j/m}C_{k/m}(\delta^{k}_{n+j}+\delta^{n}_{k+j}-\delta^{j}_{k+n})+\dot{a}_{20}\sum_{i=1}^{N}\sum_{j=1}^{N}b_{2j/m}C_{k/m}(+\delta^{k}_{n+j}+\delta^{n}_{k+j}$$

$$-\delta^{j}_{k+n})+\dot{a}_{20}\sum_{i=1}^{N}\sum_{k=1}^{N}c_{2i/m}B_{k/m}(\delta^{i}_{k+n}-\delta^{k}_{n+i}+\delta^{n}_{k+i})+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}B_{j/m}C_{k/m}(\delta^{j+n}_{i+k}-\delta^{j}_{i+k+n}+\delta^{n}_{i+j+k}-$$

$$\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}-\delta^{k+n}_{i+j}+\delta^{k}_{n+i+j})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}B_{j/m}B_{k/m}(\delta^{j+n}_{i+k}-\delta^{j}_{i+k+n}+\delta^{n}_{i+j+k}+\delta^{i}_{n+j+k}-\delta^{k+j}_{n+i}+$$

$$\delta^{k+n}_{i+j}-\delta^{k}_{n+i+j})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}C_{j/m}C_{k/m}(\delta^{j+n}_{i+k}-\delta^{j}_{i+k+n}-\delta^{n}_{i+j+k}-\delta^{i}_{n+j+k}+\delta^{k+j}_{n+i}+\delta^{k+n}_{i+j}-\delta^{k}_{n+i+j})$$

$$f_3^4 = 2\dot{a}_{20}\dot{a}_{10}Q_{n/m} + \dot{a}_{10}^2 C_{n/m} + \dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} P_{j/m}Q_{k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) + \dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} B_{j/m}Q_{k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} -$$

$$\delta_{k+n}^{j}) + \dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{j/m}P_{k/m}(\delta_{k+n}^{j} - \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} B_{i/m}P_{j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} -$$

$$\delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{i/m}P_{j/m}P_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i} - \delta_{n+i}^{k+j} +$$

$$\delta_{i+j}^{k+n} - \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{i/m}Q_{j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k})$$

$$f_3^5 = a_{20}\dot{a}_{20}Q_{n/m} + a_{20}\dot{a}_{10}C_{n/m} + \frac{1}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} B_{j/m}Q_{k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) + \frac{1}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{j/m}P_{k/m}(\delta_{k+n}^{j} -$$

$$\delta_{n+j}^{k} + \delta_{k+j}^{n}) + \dot{a}_{20}\dot{a}_{10}c_{2n} + \frac{1}{2}\dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2\,j/m}Q_{k/m}(+\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) + \frac{1}{2}\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2\,j/m}C_{k/m}(\delta_{n+j}^{k} +$$

$$\delta_{k+j}^{n} - \delta_{k+n}^{j}) + \frac{1}{2}\dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2\,j/m}P_{k/m}(\delta_{k+n}^{j} - \delta_{n+j}^{k} - \delta_{k+j}^{n}) + \frac{1}{2}\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2\,j/m}B_{k/m}(\delta_{k+n}^{j} - \delta_{n+j}^{k} + \delta_{k+j}^{n}) +$$

$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}B_{j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}$$

$$C_{j/m}P_{k/m}(\delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k} - \delta_{i+k}^{j+n}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}B_{j/m}P_{k/m}(\delta_{i+k}^{j+n} -$$

$$\delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i} - \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}C_{j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} -$$

$$\delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k})$$

$$f_3^6 = a_{20}^2 C_{n/m} + 2a_{20}\dot{a}_{20}c_{2n/m} + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2\,j/m}B_{k/m}(\delta_{k+n}^{j} - \delta_{n+j}^{k} + \delta_{k+j}^{n}) + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2\,j/m}C_{k/m}(\delta_{n+j}^{k} +$$

$$\delta_{k+j}^{n} - \delta_{k+n}^{j}) + \dot{a}_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2\,j/m}b_{2k/m}(\delta_{k+n}^{j} - \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}b_{2\,j/m}C_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} +$$

$$\delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}c_{2\,j/m}B_{k/m}(\delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j}$$

$$+ \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k} - \delta_{i+k}^{j+n}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}c_{2\,j/m}C_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n}$$

$$- \delta_{n+i+j}^{k})$$

$$f_3^7 = \dot{c}_{2n/m} - \frac{n\Omega}{m}b_{2n/m}$$

$$f_3^8 = 3\dot{a}_{10}^2 Q_{n/m} + 3\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} P_{j/m}Q_{k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) + \frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} P_{i/m}Q_{j/m}P_{k/m}(\delta_{i+k+n}^{j} + \delta_{i+j+k}^{n}$$

$$- \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k} - \delta_{i+k}^{j+n}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} Q_{i/m}Q_{j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n}$$

$$- \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k})$$

$$f_3^9 = 2\dot{a}_{10}a_{20}Q_{n/m} + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} P_{j/m}Q_{k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) + \dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m}Q_{k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) +$$

$$\dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2j/m}P_{k/m}(\delta_{k+n}^{j} - \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \dot{a}_{10}^2 c_{2n/m} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}P_{j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} +$$

$$\delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}P_{j/m}P_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} + \delta_{i+j+k}^{n} + \delta_{n+j+k}^{i}$$

$$- \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}Q_{j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n}$$

$$- \delta_{n+i+j}^{k})$$

$$f_3^{10} = a_{20}^2 Q_{n/m} + 2\dot{a}_{10}a_{20}c_{2n/m} + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m}Q_{k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) + a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2j/m}P_{k/m}(\delta_{k+n}^{j}$$

$$- \delta_{n+j}^{k} + \delta_{k+j}^{n}) + \dot{a}_{10}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m}c_{2k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}c_{2j/m}P_{k/m}(\delta_{i+k+n}^{j}$$

$$+ \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k} - \delta_{i+k}^{j+n}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}b_{2j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} +$$

$$\delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}c_{2j/m}Q_{k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} -$$

$$\delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k})$$

$$f_3^{12} = 3a_{20}^2 c_{2n/m} + 3a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2j/m}c_{2k/m}(\delta_{n+j}^{k} + \delta_{k+j}^{n} - \delta_{k+n}^{j}) + \frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} b_{2i/m}c_{2j/m}b_{2k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} +$$

$$\delta_{i+j+k}^{n} - \delta_{n+j+k}^{i} + \delta_{n+i}^{k+j} - \delta_{i+j}^{k+n} + \delta_{n+i+j}^{k}) + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{2i/m}c_{2j/m}c_{2k/m}(\delta_{i+k}^{j+n} - \delta_{i+k+n}^{j} - \delta_{i+j+k}^{n} - \delta_{n+j+k}^{i}$$

$$+ \delta_{n+i}^{k+j} + \delta_{i+j}^{k+n} - \delta_{n+i+j}^{k})$$

$$f_3^{11} = \dot{c}_{1n/m} - \frac{n\Omega}{m}b_{1n/m},\; f_3^{13} = c_{2n/m},\; f_{31}^{14} = \dot{c}_{1n/m} - \frac{n\Omega}{m}b_{1n/m},\; f_{31}^{15} = c_{1n/m},\; f_{32}^{14} = \dot{c}_{2n/m} - \frac{n\Omega}{m}b_{2n/m},$$

$$f_{32}^{15} = c_{2n/m}$$

A.4 Additional terms for nonlinear cable structure

$$f_{11}^{16}=a_{10}^{3}+\frac{3}{2}a_{10}\sum_{i=1}^{N}(b_{1i/m}^{2}+c_{1i/m}^{2})+\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{1i/m}c_{1j/m}b_{1k/m}(\delta_{j+k}^{i}-\delta_{i+j}^{k}+\delta_{i+k}^{j})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}$$
$$b_{1i/m}b_{1j/m}b_{1k/m}(\delta_{i+j}^{k}+\delta_{i+k}^{j}+\delta_{j+k}^{i})$$

$$f_{12}^{16}=a_{20}^{3}+\frac{3}{2}a_{20}\sum_{i=1}^{N}(b_{2i/m}^{2}+c_{2i/m}^{2})+\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}c_{2j/m}b_{2k/m}(\delta_{j+k}^{i}-\delta_{i+j}^{k}+\delta_{i+k}^{j})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}$$
$$b_{2i/m}b_{2j/m}b_{2k/m}(\delta_{i+j}^{k}+\delta_{i+k}^{j}+\delta_{j+k}^{i})$$

$$f_{21}^{16}=3a_{10}^{2}b_{1n/m}+\frac{3}{2}a_{10}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{1j/m}b_{1k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}+\delta_{k+j}^{n})+\frac{3}{2}a_{10}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{1j/m}c_{1k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}$$
$$-\delta_{k+j}^{n})+\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{1i/m}c_{1j/m}b_{1k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}+\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}-\delta_{i+j}^{k+n}-\delta_{n+i+j}^{k})+$$
$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{1i/m}b_{1j/m}b_{1k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}+\delta_{i+j+k}^{n}+\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})$$

$$f_{22}^{16}=3a_{20}^{2}b_{2n/m}+\frac{3}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2j/m}b_{2k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}+\delta_{k+j}^{n})+\frac{3}{2}a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2j/m}c_{2k/m}(\delta_{k+n}^{j}+\delta_{n+j}^{k}$$
$$-\delta_{k+j}^{n})+\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}c_{2j/m}b_{2k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}+\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}-\delta_{i+j}^{k+n}-\delta_{n+i+j}^{k})+$$
$$\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}b_{2j/m}b_{2k/m}(\delta_{i+k}^{j+n}+\delta_{i+k+n}^{j}+\delta_{i+j+k}^{n}+\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})$$

$$f_{31}^{16}=3a_{10}^{2}c_{1n/m}+3a_{10}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{1j/m}c_{1k/m}(\delta_{n+j}^{k}+\delta_{k+j}^{n}-\delta_{k+n}^{j})+\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{1i/m}c_{1j/m}b_{1k/m}(\delta_{i+k}^{j+n}-\delta_{i+k+n}^{j}+$$
$$\delta_{i+j+k}^{n}-\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}-\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{1i/m}c_{1j/m}c_{1k/m}(\delta_{i+k}^{j+n}-\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}-\delta_{n+j+k}^{i}$$
$$+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}-\delta_{n+i+j}^{k})$$

$$f_{32}^{16}=3a_{20}^{2}c_{2n/m}+3a_{20}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2j/m}c_{2k/m}(\delta_{n+j}^{k}+\delta_{k+j}^{n}-\delta_{k+n}^{j})+\frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}b_{2i/m}c_{2j/m}b_{2k/m}(\delta_{i+k}^{j+n}-\delta_{i+k+n}^{j}+$$
$$\delta_{i+j+k}^{n}-\delta_{n+j+k}^{i}+\delta_{n+i}^{k+j}-\delta_{i+j}^{k+n}+\delta_{n+i+j}^{k})+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}c_{2i/m}c_{2j/m}c_{2k/m}(\delta_{i+k}^{j+n}-\delta_{i+k+n}^{j}-\delta_{i+j+k}^{n}-\delta_{n+j+k}^{i}$$
$$+\delta_{n+i}^{k+j}+\delta_{i+j}^{k+n}-\delta_{n+i+j}^{k})$$

APPENDIX B

B.1 Mathematical expression for the derivative of the constant term.

$$\frac{\partial f_1^3}{\partial a_{20}} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}^2 + C_{i/m}^2) + \dot{a}_{20}^2,\ \frac{\partial f_1^5}{\partial a_{20}} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}P_{i/m} + C_{i/m}Q_{i/m}) + \dot{a}_{20}\dot{a}_{10},$$

$$\frac{\partial f_1^6}{\partial a_{20}} = \sum_{i=1}^{N}(b_{2i/m}B_{i/m} + c_{2i/m}C_{i/m}) + 2a_{20}\dot{a}_{20},\ \frac{\partial f_1^9}{\partial a_{20}} = \frac{1}{2}\sum_{i=1}^{N}(P_{i/m}^2 + Q_{i/m}^2) + \dot{a}_{10}^2,$$

$$\frac{\partial f_1^{10}}{\partial a_{20}} = \sum_{i=1}^{N}(b_{2i/m}P_{i/m} + c_{2i/m}Q_{i/m}) + 2a_{20}\dot{a}_{10},\ \frac{\partial f_1^{12}}{\partial a_{20}} = 3a_{20}^2 + \frac{3}{2}\sum_{i=1}^{N}(b_{2i/m}^2 + c_{2i/m}^2),$$

$$\frac{\partial f_1^2}{\partial Q_n} = \sum_{i=1}^{N}(B_{i/m}C_{(i+n)/m} + B_{i/m}C_{(n-i)/m} - B_{i/m}C_{(i-n)/m})$$

$$\frac{\partial f_1^4}{\partial Q_n} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}Q_{(i+n)/m} + B_{i/m}Q_{(n-i)/m} - B_{i/m}Q_{(i-n)/m}) + \frac{1}{2}\sum_{i=1}^{N}(C_{i/m}P_{(i-n)/m} + C_{i/m}P_{(n-i)/m} - C_{i/m}P_{(i+n)/m})$$

$$\frac{\partial f_1^5}{\partial Q_n} = \frac{1}{2}a_{20}C_{n/m} + \frac{1}{4}\sum_{i=1}^{N}(b_{2i/m}C_{(i+n)/m} + b_{2i/m}C_{(n-i)/m} - b_{2i/m}C_{(i-n)/m}) + \frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}B_{(n-i)/m} + c_{2i/m}B_{(i-n)/m} - c_{2i/m}B_{(i+n)/m})$$

$$\frac{\partial f_1^8}{\partial Q_n} = \frac{3}{4}\sum_{i=1}^{N}(P_{(i+n)/m}Q_{i/m} + P_{(n-i)/m}Q_{i/m} - P_{(i-n)/m}Q_{i/m})$$

$$\frac{\partial f_1^9}{\partial Q_n} = a_{20}Q_{n/m} + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}Q_{(i+n)/m} + b_{2i/m}Q_{(n-i)/m} - b_{2i/m}Q_{(i-n)/m}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}P_{(n-i)/m} + c_{2i/m}P_{(i-n)/m} - c_{2i/m}P_{(i+n)/m})$$

$$\frac{\partial f_1^{10}}{\partial Q_n} = a_{20}c_{2n/m} + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}c_{2(i+n)/m} + b_{2i/m}c_{2(n-i)/m} - b_{2i/m}c_{2(i-n)/m})$$

$$\frac{\partial f_1^2}{\partial P_n} = \frac{1}{4}\sum_{j=1}^{N}(B_{j/m}B_{(j+n)/m} + B_{j/m}B_{(j-n)/m} + B_{j/m}B_{(n-j)/m}) + \frac{1}{4}\sum_{j=1}^{N}(C_{j/m}C_{(j+n)/m} + C_{j/m}C_{(j-n)/m} - C_{j/m}C_{(n-j)/m})$$

$$\frac{\partial f_1^4}{\partial P_n} = \frac{1}{2}\sum_{j=1}^{N}(B_{j/m}P_{(j+n)/m} + B_{j/m}P_{(j-n)/m} + B_{j/m}P_{(n-j)/m}) + \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}Q_{(j+n)/m} + C_{j/m}Q_{(j-n)/m} - C_{j/m}Q_{(n-j)/m})$$

$$\frac{\partial f_1^5}{\partial P_n} = \frac{1}{2}a_{20}B_{n/m} + \frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}B_{(j+n)/m} + b_{2j/m}B_{(j-n)/m} + b_{2j/m}B_{(n-j)/m}) + \frac{1}{4}\sum_{j=1}^{N}(c_{2j/m}C_{(j+n)/m} + c_{2j/m}C_{(j-n)/m} - c_{2j/m}C_{(n-j)/m})$$

$$\frac{\partial f_1^8}{\partial P_n} = \frac{3}{4}\sum_{j=1}^{N}(P_{j/m}P_{(j+n)/m} + P_{j/m}P_{(j-n)/m} + P_{j/m}P_{(n-j)/m}) + \frac{3}{4}\sum_{j=1}^{N}(Q_{j/m}Q_{(j+n)/m} + Q_{j/m}Q_{(j-n)/m}$$

$$-Q_{j/m}Q_{(n-j)/m})$$

$$\frac{\partial f_1^9}{\partial P_n}=a_{20}P_{n/m}+\frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}P_{(j+n)/m}+b_{2j/m}P_{(j-n)/m}+b_{2j/m}P_{(n-j)/m})+\frac{1}{2}\sum_{j=1}^{N}(c_{2j/m}Q_{(j+n)/m}+c_{2j/m}Q_{(j-n)/m}$$
$$-c_{2j/m}Q_{(n-j)/m})$$

$$\frac{\partial f_1^{10}}{\partial P_n}=a_{20}b_{2n/m}+\frac{1}{4}\sum_{i=1}^{N}(b_{2i/m}b_{2(i+n)/m}+b_{2i/m}b_{2(i-n)/m}+b_{2i/m}b_{2(n-i)/m})+\frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}c_{2(i+n)/m}+c_{2i/m}c_{2(i-n)/m}$$
$$-c_{2i/m}c_{2(n-i)/m})$$

$$\frac{\partial f_1^1}{\partial C_n}=\frac{3}{2}\sum_{j=1}^{N}(B_{(j-n)/m}C_{j/m}+B_{(n-j)/m}C_{j/m}-B_{(n+j)/m}C_{j/m})$$

$$\frac{\partial f_1^2}{\partial C_n}=\frac{1}{2}\sum_{j=1}^{N}(C_{j/m}P_{(j-n)/m}+C_{j/m}P_{(n-j)/m}-C_{j/m}P_{(j+n)/m})+\frac{1}{2}\sum_{i=1}^{N}(B_{i/m}Q_{(i+n)/m}+B_{i/m}Q_{(n-i)/m}-B_{i/m}Q_{(i-n)/m})$$

$$\frac{\partial f_1^3}{\partial C_n}=a_{20}C_{n/m}+\frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}C_{(i+n)/m}+b_{2i}C_{(n-i)/m}-b_{2i}C_{(i-n)/m})+\frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}B_{(n-i)/m}+c_{2i/m}B_{(i-n)/m}-$$
$$c_{2i/m}B_{(i+n)/m})$$

$$\frac{\partial f_1^4}{\partial C_n}=\frac{1}{2}\sum_{j=1}^{N}(P_{j/m}Q_{(j+n)/m}+P_{j/m}Q_{(n-j)/m}-P_{j/m}Q_{(j-n)/m})$$

$$\frac{\partial f_1^5}{\partial C_n}=a_{20}Q_{n/m}+\frac{1}{4}\sum_{i=1}^{N}(b_{2i/m}Q_{(i+n)/m}+b_{2i/m}Q_{(n-i)/m}-b_{2i/m}Q_{(i-n)/m})+\frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}P_{(n-i)/m}+c_{2i/m}P_{(i-n)/m}-$$
$$c_{2i/m}P_{(i+n)/m})$$

$$\frac{\partial f_1^6}{\partial C_n}=a_{20}c_{2n/m}+\frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}c_{2(i+n)/m}+b_{2i/m}c_{2(n-i)/m}-b_{2i/m}c_{2(i-n)/m})]$$

$$\frac{\partial f_1^3}{\partial b_{2n}}=\frac{1}{4}\sum_{j=1}^{N}(B_{j/m}B_{(j+n)/m}+B_{j/m}B_{(j-n)m}+B_{j/m}B_{(n-j)/m})+\frac{1}{4}\sum_{j=1}^{N}(C_{j/m}C_{(j+n)/m}+C_{j/m}C_{(j-n)/m}-$$
$$C_{j/m}C_{(n-j)/m})$$

$$\frac{\partial f_1^5}{\partial b_{2n}}=\frac{1}{4}\sum_{j=1}^{N}(B_{j/m}P_{(j+n)/m}+B_{j/m}P_{(j-n)/m}+B_{j/m}P_{(n-j)/m})+\frac{1}{4}\sum_{j=1}^{N}(C_{j/m}Q_{(j+n)/m}+C_{j/m}Q_{(j-n)/m}-$$
$$C_{j/m}Q_{(n-j)/m})$$

$$\frac{\partial f_1^6}{\partial b_{2n}}=a_{20}B_n+\frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}B_{(j+n)/m}+b_{2j/m}B_{(j-n)/m}+b_{2j/m}B_{(n-j)/m})+\frac{1}{2}\sum_{j=1}^{N}(c_{2j/m}C_{(j+n)/m}+c_{2j/m}C_{(j-n)/m}$$
$$-c_{2j/m}C_{(n-j)/m})$$

$$\frac{\partial f_1^9}{\partial b_{2n}}=\frac{1}{4}\sum_{j=1}^{N}(P_{j/m}P_{(j+n)/m}+P_{j/m}P_{(j-n)/m}+P_{j/m}P_{(n-j)/m})+\frac{1}{4}\sum_{j=1}^{N}(Q_{j/m}Q_{(j+n)/m}+Q_{j/m}Q_{(j-n)/m}$$
$$-Q_{j/m}Q_{(n-j)/m})$$

$$\frac{\partial f_1^{10}}{\partial b_{2n}} = a_{20}P_{n/m} + \frac{1}{2}\sum_{j=1}^{N}(c_{2j/m}Q_{(j+n)/m} + c_{2j/m}Q_{(j-n)/m} - c_{2j/m}Q_{(n-j)/m}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}P_{(j+n)/m} + b_{2j/m}P_{(j-n)/m} + b_{2j/m}P_{(n-j)/m})$$

$$\frac{\partial f_1^{12}}{\partial b_{2n}} = 3a_{20}b_{2n/m} + \frac{3}{4}\sum_{j=1}^{N}(b_{2j/m}b_{2(j+n)/m} + b_{2j/m}b_{2(n-j)/m} + b_{2j/m}b_{2(j-n)/m}) + \frac{3}{4}\sum_{i=1}^{N}(c_{2i/m}c_{2(i+n)/m} + c_{2i/m}c_{2(i-n)/m} - c_{2i/m}c_{2(n-i)/m})$$

$$\frac{\partial f_1^{1}}{\partial B_{n}} = \frac{3}{4}\sum_{j=1}^{N}(B_{(j-n)/m}B_{j/m} + B_{(n-j)/m}B_{j/m} + B_{(n+j)/m}B_{j/m}) + \frac{3}{4}\sum_{j=1}^{N}(C_{j/m}C_{(j+n)/m} + C_{j/m}C_{(j-n)/m} - C_{j/m}C_{(n-j)/m})$$

$$\frac{\partial f_1^{2}}{\partial B_{n}} = \frac{1}{2}\sum_{j=1}^{N}(B_{j/m}P_{(j+n)/m} + B_{j/m}P_{(j-n)/m} + B_{j/m}P_{(n-j)/m}) + \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}Q_{(j+n)/m} + C_{j/m}Q_{(j-n)/m} - C_{j/m}Q_{(n-j)/m})$$

$$\frac{\partial f_1^{3}}{\partial B_{n}} = a_{20}B_{n/m} + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}B_{(j+n)/m} + b_{2j/m}B_{(j-n)/m} + b_{2j/m}B_{(n-j)/m}) + \frac{1}{2}\sum_{j=1}^{N}(c_{2j/m}C_{(j+n)/m} + c_{2j/m}C_{(j-n)/m} - c_{2j/m}C_{(n-j)/m})$$

$$\frac{\partial f_1^{4}}{\partial B_{n}} = \frac{1}{4}\sum_{j=1}^{N}(P_{j/m}P_{(j+n)/m} + P_{j/m}P_{(j-n)/m} + P_{j/m}P_{(n-j)/m}) + \frac{1}{4}\sum_{j=1}^{N}(Q_{j/m}Q_{(j+n)/m} + Q_{j/m}Q_{(j-n)/m} - Q_{j/m}Q_{(n-j)/m})$$

$$\frac{\partial f_1^{5}}{\partial B_{n}} = \frac{a_{20}}{2}P_{n/m} + \frac{1}{4}\sum_{j=1}^{N}(c_{2j/m}Q_{(j+n)/m} + c_{2j/m}Q_{(j-n)/m} - c_{2j/m}Q_{(n-j)/m}) + \frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}P_{(j+n)/m} + b_{2j/m}P_{(j-n)/m} + b_{2j/m}P_{(n-j)/m})$$

$$\frac{\partial f_1^{6}}{\partial B_{n}} = a_{20}b_{2n/m} + \frac{1}{4}\sum_{i=1}^{N}(b_{2i/m}b_{2(i+n)/m} + b_{2i/m}b_{2(i-n)/m} + b_{2i/m}b_{2(n-i)/m}) + \frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}c_{2(i+n)/m} + c_{2i/m}c_{2(i-n)/m} - c_{2i/m}c_{2(n-i)/m})$$

$$\frac{\partial f_1^{3}}{\partial c_{2n}} = \frac{1}{2}\sum_{j=1}^{N}(B_{(j-n)/m}C_{j/m} + B_{(n-j)/m}C_{j/m} - B_{(n+j)/m}C_{j/m})$$

$$\frac{\partial f_1^{5}}{\partial c_{2n}} = \frac{1}{4}\sum_{j=1}^{N}(C_{j/m}P_{(j-n)/m} + C_{j/m}P_{(n-j)/m} - C_{j/m}P_{(j+n)/m}) + \frac{1}{4}\sum_{j=1}^{N}(B_{j/m}Q_{(n+j)/m} + B_{j/m}Q_{(n-j)/m} - B_{j/m}Q_{(j-n)/m})$$

$$\frac{\partial f_1^{6}}{\partial c_{2n}} = a_{20}C_{n/m} + \sum_{i=1}^{N}(c_{2i/m}B_{(n-i)/m} + c_{2i/m}B_{(i-n)/m} - c_{2i/m}B_{(i+n)/m}) + \sum_{i=1}^{N}(b_{2i/m}C_{(i+n)/m} + b_{2i/m}C_{(n-i)/m} - b_{2i/m}C_{(i-n)/m})$$

$$\frac{\partial f_1^{9}}{\partial c_{2n}} = \frac{1}{2}\sum_{j=1}^{N}(P_{j/m}Q_{(j+n)/m} + P_{j/m}Q_{(n-j)/m} - P_{j/m}Q_{(j-n)/m})$$

$$\frac{\partial f_1^{10}}{\partial c_{2n}} = a_{20}Q_{n/m} + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}Q_{(i+n)/m} + b_{2i/m}Q_{(n-i)/m} - b_{2i/m}Q_{(i-n)/m}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}P_{(n-i)/m} + c_{2i/m}P_{(i-n)/m} - c_{2i/m}P_{(i+n)/m})$$

$$\frac{\partial f_1^{12}}{\partial c_{2n}} = 3a_{20}c_{2n/m} + \frac{3}{2}\sum_{i=1}^{N}(b_{2i/m}c_{2(i+n)/m} + b_{2i/m}c_{2(n-i)/m} - b_{2i/m}c_{2(i-n)/m})$$

$$\frac{\partial C_n}{\partial b_{2n}} = (-\frac{n\Omega}{m}),\ \frac{\partial P_n}{\partial c_{1n}} = (\frac{n\Omega}{m}),\ \frac{\partial Q_n}{\partial b_{1n}} = (-\frac{n\Omega}{m}),\ \frac{\partial B_n}{\partial c_{2n}} = (\frac{n\Omega}{m})$$

B.2 Mathematical expression for the derivative of the cosine term.

$$\frac{\partial f_2^{3}}{\partial a_{20}} = \frac{1}{2}\sum_{j=1}^{N}(B_{j-k}B_{j/m} + B_{k-j}B_{j/m} + B_{k+j}B_{j/m}) + \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}C_{j+k} + C_{j/m}C_{j-k} - C_{j/m}C_{k-j})$$

$$\frac{\partial f_2^{5}}{\partial a_{20}} = \frac{1}{2}\sum_{j=1}^{N}(B_{j/m}P_{j+k} + B_{j/m}P_{j-k} + B_{j/m}P_{k-j}) + \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}Q_{j+k} + C_{j/m}Q_{j-k} - C_{j/m}Q_{k-j})$$

$$\frac{\partial f_2^{6}}{\partial a_{20}} = 2a_{20}B_{k/m} + \sum_{j=1}^{N}(c_{2j/m}C_{j+k} + c_{2j/m}C_{j-k} - c_{2j/m}C_{k-j}) + \sum_{j=1}^{N}(b_{2j/m}B_{j+k} + b_{2j/m}B_{j-k} + b_{2j/m}B_{k-j})$$

$$\frac{\partial f_2^{9}}{\partial a_{20}} = \frac{1}{2}\sum_{j=1}^{N}(P_{j/m}P_{j+k} + P_{j/m}P_{j-k} + P_{j/m}P_{k-j}) + \frac{1}{2}\sum_{j=1}^{N}(Q_{j/m}Q_{j+k} + Q_{j/m}Q_{j-k} - Q_{j/m}Q_{k-j})$$

$$\frac{\partial f_2^{10}}{\partial a_{20}} = 2a_{20}P_{k/m} + \sum_{j=1}^{N}(b_{2j/m}P_{j+k} + b_{2j/m}P_{j-k} + b_{2j/m}P_{k-j}) + \sum_{j=1}^{N}(c_{2j/m}Q_{j+k} + c_{2j/m}Q_{j-k} - c_{2j/m}Q_{k-j})$$

$$\frac{\partial f_2^{12}}{\partial a_{20}} = 6a_{20}b_{2k/m} + \frac{3}{2}\sum_{i=1}^{N}(b_{2i/m}b_{2(i+k)} + b_{2i/m}b_{2(i-k)} + b_{2i/m}b_{2(k-i)}) + \frac{3}{2}\sum_{i=1}^{N}(c_{2i/m}c_{2(i+k)} + c_{2i/m}c_{2(i-k)} - c_{2i/m}c_{2(k-i)})$$

$$\frac{\partial f_2^{2}}{\partial Q_n} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}C_{i+r+k} + B_{i/m}C_{r-i-k} + B_{i/m}C_{r+k-i} + B_{i/m}C_{i+r-k} - B_{i/m}C_{k-i-r} - B_{i/m}C_{i-r-k} - B_{i/m}C_{k+i-r})$$

$$\frac{\partial f_2^{4}}{\partial Q_n} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}Q_{i+r+k} + B_{i/m}Q_{r-i-k} + B_{i/m}Q_{r+k-i} + B_{i/m}Q_{i+r-k} - B_{i/m}Q_{k-i-r} - B_{i/m}Q_{i-r-k} - B_{i/m}Q_{k+i-r}) + \frac{1}{2}\sum_{i=1}^{N}(C_{i/m}P_{k+i-r} + C_{i/m}P_{i-r-k} + C_{i/m}P_{r-i-k} + C_{i/m}P_{r+k-i} - C_{i/m}P_{i+r-k} - C_{i/m}P_{k-i-r} - C_{i/m}P_{i+r+k})$$

$$\frac{\partial f_2^{5}}{\partial Q_n} = \frac{1}{2}a_{20}(C_{r-k} + C_{r+k} - C_{k-r}) + \frac{1}{4}\sum_{i=1}^{N}(b_{2i/m}C_{i+r+k} + b_{2i/m}C_{r-i-k} + b_{2i/m}C_{r+k-i} + b_{2i/m}C_{i+r-k} - b_{2i/m}C_{k-i-r} - b_{2i/m}C_{i-r-k} - b_{2i/m}C_{k+i-r}) + \frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}B_{k+i-r} + c_{2i/m}B_{i-r-k} + c_{2i/m}B_{r-i-k} + c_{2i/m}B_{r+k-i} - c_{2i/m}B_{i+r-k} - c_{2i/m}B_{k-i-r} - c_{2i/m}B_{i+r+k})$$

$$\frac{\partial f_2^8}{\partial Q_n} = \frac{3}{2}\sum_{j=1}^{N}(Q_{j/m}P_{j-r-k} + Q_{j/m}P_{j+k-r} + Q_{j/m}P_{r+k-j} + Q_{j/m}P_{r-k-j} - Q_{j/m}P_{k-r-j} - Q_{j/m}P_{r+k+j} - Q_{j/m}P_{r+j-k})$$

$$\frac{\partial f_2^9}{\partial Q_n} = a_{20}(Q_{r-k} + Q_{r+k} - Q_{k-r}) + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}Q_{i+r+k} + b_{2i/m}Q_{r-i-k} + b_{2i/m}Q_{r+k-i} + b_{2i/m}Q_{i+r-k} - b_{2i/m}Q_{k-i-r}$$
$$- b_{2i/m}Q_{i-r-k} - b_{2i/m}Q_{k+i-r}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}P_{k+i-r} + c_{2i/m}P_{i-r-k} + c_{2i/m}P_{r-i-k} + c_{2i/m}P_{r+k-i} - c_{2i/m}P_{i+r-k}$$
$$- c_{2i/m}P_{k-i-r} - c_{2i/m}P_{i+r+k})$$

$$\frac{\partial f_2^{10}}{\partial Q_n} = a_{20}(c_{2(r-k)} + c_{2(r+k)} - c_{2(k-r)}) + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}c_{2(i+r+k)} + b_{2i/m}c_{2(r-i-k)} + b_{2i/m}c_{2(r+k-i)} + b_{2i/m}c_{2(i+r-k)}$$
$$- b_{2i/m}c_{2(k-i-r)} - b_{2i/m}c_{2(i-r-k)} - b_{2i/m}c_{2(k+i-r)})$$

$$\frac{\partial f_2^2}{\partial P_n} = \frac{1}{4}\sum_{j=1}^{N}(B_{j/m}B_{j-n-k} + B_{j/m}B_{j+k-n} + B_{j/m}B_{n+k-j} + B_{j/m}B_{n-k-j} + B_{j/m}B_{k-n-j} + B_{j/m}B_{n+k+j} +$$
$$B_{j/m}B_{n+j-k}) + \frac{1}{4}\sum_{j=1}^{N}(C_{j/m}C_{j-n-k} + C_{j/m}C_{j+k-n} - C_{j/m}C_{n+k-j} - C_{j/m}C_{n-k-j} - C_{j/m}C_{k-n-j} +$$
$$C_{j/m}C_{n+k+j} + C_{j/m}C_{n+j-k})$$

$$\frac{\partial f_2^4}{\partial P_n} = \frac{1}{2}\sum_{j=1}^{N}(B_{j/m}P_{j-n-k} + B_{j/m}P_{j+k-n} + B_{j/m}P_{n+k-j} + B_{j/m}P_{n-k-j} + B_{j/m}P_{k-n-j} + B_{j/m}P_{n+k+j} + B_{j/m}P_{n+j-k})$$
$$+ \frac{1}{2}\sum_{i=1}^{N}(C_{i/m}Q_{k+i-n} + C_{i/m}Q_{i-n-k} + C_{i/m}Q_{i+n+k} + C_{i/m}Q_{i+n-k} - C_{i/m}Q_{n-i-k} - C_{i/m}Q_{n+k-i} - C_{i/m}Q_{k-i-n})$$

$$\frac{\partial f_2^5}{\partial P_n} = [\frac{1}{2}a_{20}(B_{n-k} + B_{n+k} + B_{k-n}) + \frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}B_{j-n-k} + b_{2j/m}B_{j+k-n} + b_{2j/m}B_{n+k-j} + b_{2j/m}B_{n-k-j} +$$
$$b_{2j/m}B_{k-n-j} + b_{2j/m}B_{n+k+j} + b_{2j/m}B_{n+j-k}) + \frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}C_{k+i-n} + c_{2i/m}C_{i-n-k} + c_{2i/m}C_{i+n+k} +$$
$$c_{2i/m}C_{i+n-k} + c_{2i/m}C_{n-i-k} + c_{2i/m}C_{n+k-i} + c_{2i/m}C_{k-i-n})$$

$$\frac{\partial f_2^8}{\partial P_n} = \frac{3}{4}\sum_{j=1}^{N}(P_{j/m}P_{j-n-k} + P_{j/m}P_{j+k-n} + P_{j/m}P_{n+k-j} + P_{j/m}P_{n-k-j} + P_{j/m}P_{k-n-j} + P_{j/m}P_{n+k+j} + P_{j/m}P_{n+j-k})$$
$$+ \frac{3}{4}\sum_{i=1}^{N}(Q_{i/m}Q_{k+i-n} + Q_{i/m}Q_{i-n-k} + Q_{i/m}Q_{i+n+k} + Q_{i/m}Q_{i+n-k} - Q_{i/m}Q_{n-i-k} - Q_{i/m}Q_{n+k-i} - Q_{i/m}Q_{k-i-n})$$

$$\frac{\partial f_2^9}{\partial P_n} = \frac{1}{2}\sum_{j=1}^{N}(B_{j/m}P_{j-n-k} + B_{j/m}P_{j+k-n} + B_{j/m}P_{n+k-j} + B_{j/m}P_{n-k-j} + B_{j/m}P_{k-n-j} + B_{j/m}P_{n+k+j} + B_{j/m}P_{n+j-k})$$
$$+ \frac{1}{2}\sum_{i=1}^{N}(C_{i/m}Q_{k+i-n} + C_{i/m}Q_{i-n-k} + C_{i/m}Q_{i+n+k} + C_{i/m}Q_{i+n-k} - C_{i/m}Q_{n-i-k} - C_{i/m}Q_{n+k-i} - C_{i/m}Q_{k-i-n})$$

$$\frac{\partial f_2^{10}}{\partial P_n} = a_{20}(b_{2(n-k)} + b_{2(n+k)} + b_{2(k-n)}) + \frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}b_{2(j-n-k)} + b_{2j/m}b_{2(j+k-n)} + b_{2j/m}b_{2(n+k-j)} + b_{2j/m}b_{2(n-k-j)}$$

$$+ b_{2j/m}b_{2(k-n-j)} + b_{2j/m}b_{2(n+k+j)} + b_{2j/m}b_{2(n+j-k}) + \frac{1}{4}\sum_{j=1}^{N}(c_{2j/m}c_{2(j-n-k)} + c_{2j/m}c_{2(j+k-n)} - c_{2j/m}c_{2(n+k-j)}$$

$$- c_{2j/m}c_{2(n-k-j)} - c_{2j/m}c_{2(k-n-j)} + c_{2j/m}c_{2(n+k+j)} + c_{2j/m}c_{2(n+j-k)}) + a_{20}^2 k\Omega\delta_n^k$$

$$\frac{\partial f_2^1}{\partial C_n} = \frac{3}{2}\sum_{i=1}^{N}(B_{i/m}C_{i+n+k} + B_{i/m}C_{n-i-k} + B_{i/m}C_{n+k-i} + B_{i/m}C_{i+n-k} - B_{i/m}C_{k-i-n} - B_{i/m}C_{i-n-k} - B_{i/m}C_{k+i-n})$$

$$\frac{\partial f_2^2}{\partial C_n} = \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}P_{j-n-k} + C_{j/m}P_{j+k-n} + C_{j/m}P_{n+k-j} + C_{j/m}P_{n-k-j} - C_{j/m}P_{k-n-j} - C_{j/m}P_{n+k+j} - C_{j/m}P_{n+j-k})$$

$$+ \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}Q_{i+n+k} + B_{i/m}Q_{n-i-k} + B_{i/m}Q_{n+k-i} + B_{i/m}Q_{i+n-k} - B_{i/m}Q_{k-i-n} - B_{i/m}Q_{i-n-k} - B_{i/m}Q_{k+i-n})$$

$$\frac{\partial f_2^3}{\partial C_n} = a_{20}(C_{n-k} + C_{n+k} - C_{k-n}) + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}C_{i+n+k} + b_{2i/m}C_{n-i-k} + b_{2i/m}C_{n+k-i} + b_{2i/m}C_{i+n-k} - b_{2i/m}C_{k-i-n}$$

$$- b_{2i/m}C_{i-n-k} - b_{2i/m}C_{k+i-n}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}B_{k+i-n} + c_{2i/m}B_{i-n-k} + c_{2i/m}B_{n-i-k} + c_{2i/m}B_{n+k-i} - c_{2i/m}B_{i+n-k}$$

$$- c_{2i/m}B_{k-i-n} - c_{2i/m}B_{i+n+k})$$

$$\frac{\partial f_2^4}{\partial C_n} = \frac{1}{2}\sum_{i=1}^{N}(P_{i/m}Q_{k+i-n} + P_{i/m}Q_{i-n-k} + P_{i/m}Q_{n-i-k} + P_{i/m}Q_{n+k-i} - P_{i/m}Q_{i+n-k} - P_{i/m}Q_{k-i-n} - P_{i/m}Q_{i+n+k})$$

$$\frac{\partial f_2^5}{\partial C_n} = [\frac{1}{2}a_{20}(Q_{n-k} + Q_{n+k} - Q_{k-n}) + \frac{1}{4}\sum_{i=1}^{N}(b_{2i/m}Q_{i+n+k} + b_{2i/m}Q_{n-i-k} + b_{2i/m}Q_{n+k-i} + b_{2i/m}Q_{i+n-k} - b_{2i/m}Q_{k-i-n}$$

$$- b_{2i/m}Q_{i-n-k} - b_{2i/m}Q_{k+i-n}) + \frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}P_{k+i-n} + c_{2i/m}P_{i-n-k} + c_{2i/m}P_{n-i-k} + c_{2i/m}P_{n+k-i} - c_{2i/m}P_{i+n-k} -$$

$$c_{2i/m}P_{k-i-n} - c_{2i/m}P_{i+n+k})$$

$$\frac{\partial f_2^6}{\partial C_n} = a_{20}(c_{2(n-k)} + c_{2(n+k)} - c_{2(k-n)}) + \frac{1}{2}\sum_{i=1}^{N}(b_{2i}c_{2(i+n+k)} + b_{2i}c_{2(n-i-k)} + b_{2i}c_{2(n+k-i)} + b_{2i}c_{2(i+n-k)} - b_{2i}c_{2(k-i-n)}$$

$$- b_{2i}c_{2(i-n-k)} - b_{2i}c_{2(k+i-n)})$$

$$\frac{\partial f_2^3}{\partial b_{2n}} = \frac{1}{4}\sum_{j=1}^{N}(B_{j/m}B_{j-n-k} + B_{j/m}B_{j+k-n} + B_{j/m}B_{n+k-j} + B_{j/m}B_{n-k-j} + B_{j/m}B_{k-n-j} + B_{j/m}B_{n+k+j} + B_{j/m}B_{n+j-k})$$

$$+ \frac{1}{4}\sum_{j=1}^{N}(C_{j/m}C_{j-n-k} + C_{j/m}C_{j+k-n} - C_{j/m}C_{n+k-j} - C_{j/m}C_{n-k-j} - C_{j/m}C_{k-n-j} + C_{j/m}C_{n+k+j} + C_{j/m}C_{n+j-k})$$

$$\frac{\partial f_2^5}{\partial b_{2n}} = \frac{1}{4}\sum_{j=1}^{N}(B_{j/m}P_{j-n-k} + B_{j/m}P_{j+k-n} + B_{j/m}P_{n+k-j} + B_{j/m}P_{n-k-j} + B_{j/m}P_{k-n-j} + B_{j/m}P_{n+k+j} + B_{j/m}P_{n+j-k})$$

$$+ \frac{1}{4}\sum_{j=1}^{N}(C_{j/m}Q_{j-n-k} + C_{j/m}Q_{j+k-n} - C_{j/m}Q_{n+k-j} - C_{j/m}Q_{n-k-j} - C_{j/m}Q_{k-n-j} + C_{j/m}Q_{n+k+j} + C_{j/m}Q_{n+j-k})$$

$$\frac{\partial f_2^6}{\partial b_{2n}} = a_{20}(B_{n-k} + B_{n+k} + B_{k-n}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j}B_{j-n-k} + b_{2j}B_{j+k-n} + b_{2j}B_{n+k-j} + b_{2j}B_{n-k-j} + b_{2j}B_{k-n-j} + b_{2j}B_{n+k+j}$$

$$+ b_{2j} B_{n+j-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i}C_{k+i-n} + c_{2i}C_{i-n-k} + c_{2i}C_{i+n+k} + c_{2i}C_{i+n-k} - c_{2i}C_{n-i-k} - c_{2i}C_{n+k-i}$$

$$- c_{2i}C_{k-i-n})$$

$$\frac{\partial f_2^9}{\partial b_{2n}} = a_{20}(B_{n-k} + B_{n+k} + B_{k-n}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}B_{j-n-k} + b_{2j/m}B_{j+k-n} + b_{2j/m}B_{n+k-j} + b_{2j/m}B_{n-k-j} +$$

$$b_{2j/m}B_{k-n-j} + b_{2j/m}B_{n+k+j} + b_{2j/m}B_{n+j-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}C_{k+i-n} + c_{2i/m}C_{i-n-k} + c_{2i/m}C_{i+n+k} +$$

$$c_{2i/m}C_{i+n-k} - c_{2i/m}C_{n-i-k} - c_{2i/m}C_{n+k-i} - c_{2i/m}C_{k-i-n})$$

$$\frac{\partial f_2^{10}}{\partial b_{2n}} = a_{20}(P_{n-k} + P_{n+k} + P_{k-n}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}P_{j-n-k} + b_{2j/m}P_{j+k-n} + b_{2j/m}P_{n+k-j} + b_{2j/m}P_{n-k-j}$$

$$+ b_{2j/m}P_{k-n-j} + b_{2j/m}P_{n+k+j} + b_{2j/m}P_{n+j-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}Q_{k+i-n} + c_{2i/m}Q_{i-n-k} + c_{2i/m}Q_{i+n+k}$$

$$+ c_{2i/m}Q_{i+n-k} - c_{2i/m}Q_{n-i-k} - c_{2i/m}Q_{n+k-i} - c_{2i/m}Q_{k-i-n})$$

$$\frac{\partial f_2^{12}}{\partial b_{2n}} = 3a_{20}(b_{2(n-k)} + b_{2(n+k)} + b_{2(k-n)}) + \frac{3}{4}\sum_{j=1}^{N}(b_{2j/m}b_{2(j-n-k)} + b_{2j/m}b_{2(j+k-n)} + b_{2j/m}b_{2(n+k-j)}$$

$$+ b_{2j/m}b_{2(n-k-j)} + b_{2j/m}b_{2(k-n-j)} + b_{2j/m}b_{2(n+k+j)} + b_{2j/m}b_{2(n+j-k)}) + \frac{3}{4}\sum_{j=1}^{N}(c_{2j/m}c_{2(j-n-k)} + \delta_n^k 3a_{20}^2$$

$$c_{2j/m}c_{2(j+k-n)} - c_{2j/m}c_{2(n+k-j)} - c_{2j/m}c_{2(n-k-j)} - c_{2j/m}c_{2(k-n-j)} + c_{2j/m}c_{2(n+k+j)} + c_{2j/m}c_{2(n+j-k)})$$

$$\partial f_2^{13} / \partial b_{2n} = \delta_n^k, \partial f_2^{15} / \partial b_{2n} = \delta_n^k$$

$$\frac{\partial f_2^1}{\partial B_n} = \frac{3}{4}\sum_{j=1}^{N}(B_{j/m}B_{j-n-k} + B_{j/m}B_{j+n-k} + B_{j/m}B_{n+k-j} + B_{j/m}B_{n-k-j} + B_{j/m}B_{k-n-j} + B_{j/m}B_{n+k+j}$$

$$+ B_{j/m}B_{n+j-k}) + \frac{3}{4}\sum_{j=1}^{N}(C_{j/m}C_{j-n-k} + C_{j/m}C_{j+k-n} - C_{j/m}C_{n+k-j} - C_{j/m}C_{n-k-j} - C_{j/m}C_{k-n-j}$$

$$+ C_{j/m}C_{n+k+j} + C_{j/m}C_{n+j-k})$$

$$\frac{\partial f_2^2}{\partial B_n} = \frac{1}{2}\sum_{j=1}^{N}(B_{j/m}P_{j-n-k} + B_{j/m}P_{j+k-n} + B_{j/m}P_{n+k-j} + B_{j/m}P_{n-k-j} + B_{j/m}P_{k-n-j} + B_{j/m}P_{n+k+j} + B_{j/m}P_{n+j-k})$$

$$+ \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}Q_{j-n-k} + C_{j/m}Q_{j+k-n} - C_{j/m}Q_{n+k-j} - C_{j/m}Q_{n-k-j} - C_{j/m}Q_{k-n-j} + C_{j/m}Q_{n+k+j}$$

$$+ C_{j/m}Q_{n+j-k})$$

$$\frac{\partial f_2^3}{\partial B_n} = a_{20}(B_{n-k} + B_{n+k} + B_{k-n}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}B_{j-n-k} + b_{2j/m}B_{j+k-n} + b_{2j/m}B_{n+k-j} + b_{2j/m}B_{n-k-j}$$

$$+ b_{2j/m}B_{k-n-j} + b_{2j/m}B_{n+k+j} + b_{2j/m}B_{n+j-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}C_{k+i-n} + c_{2i/m}C_{i-n-k} + c_{2i/m}C_{i+n+k}$$

$$+ c_{2i/m}C_{i+n-k} - c_{2i/m}C_{n-i-k} - c_{2i/m}C_{n+k-i} - c_{2i/m}C_{k-i-n})$$

$$\frac{\partial f_2^4}{\partial B_n}=\frac{1}{4}\sum_{j=1}^{N}(P_{j/m}P_{j-n-k}+P_{j/m}P_{j+k-n}+P_{j/m}P_{n+k-j}+P_{j/m}P_{n-k-j}+P_{j/m}P_{k-n-j}+P_{j/m}P_{n+k+j}+P_{j/m}P_{n+j-k})$$
$$+\frac{1}{4}\sum_{i=1}^{N}(Q_{i/m}Q_{k+i-n}+Q_{i/m}Q_{i-n-k}+Q_{i/m}Q_{i+n+k}+Q_{i/m}Q_{i+n-k}-Q_{i/m}Q_{n-i-k}-Q_{i/m}Q_{n+k-i}-Q_{i/m}Q_{k-i-n})$$

$$\frac{\partial f_2^5}{\partial B_n}=\frac{1}{2}a_{20}(P_{n-k}+P_{n+k}+P_{k-n})+\frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}P_{j-n-k}+b_{2j/m}P_{j+k-n}+b_{2j/m}P_{n+k-j}+b_{2j/m}P_{n-k-j}$$
$$+b_{2j/m}P_{k-n-j}+b_{2j/m}P_{n+k+j}+b_{2j/m}P_{n+j-k})+\frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}Q_{k+i-n}+c_{2i/m}Q_{i-n-k}+c_{2i/m}Q_{i+n+k}$$
$$+c_{2i/m}Q_{i+n-k}-c_{2i/m}Q_{n-i-k}-c_{2i/m}Q_{n+k-i}-c_{2i/m}Q_{k-i-n})$$

$$\frac{\partial f_2^6}{\partial B_n}=a_{20}(b_{2(n-k)}+b_{2(n+k)}+b_{2(k-n)})+\frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}b_{2(j-n-k)}+b_{2j/m}b_{2(j+k-n)}+b_{2j/m}b_{2(n+k-j)}+$$
$$b_{2j/m}b_{2(n-k-j)}+b_{2j/m}b_{2(k-n-j)}+b_{2j/m}b_{2(n+k+j)}+b_{2j/m}b_{2(n+j-k)})+\frac{1}{4}\sum_{j=1}^{N}(c_{2j/m}c_{2(j-n-k)}+c_{2j/m}c_{2(j+k-n)}$$
$$-c_{2j/m}c_{2(n+k-j)}-c_{2j/m}c_{2(n-k-j)}-c_{2j/m}c_{2(k-n-j)}+c_{2j/m}c_{2(n+k+j)}+c_{2j/m}c_{2(n+j-k)})+\delta_n^k k\Omega\alpha_6 a_{20}^2$$

$$\frac{\partial f_2^3}{\partial c_{2n}}=\frac{1}{2}\sum_{i=1}^{N}(B_{i/m}C_{i+n+k}+B_{i/m}C_{n-i-k}+B_{i/m}C_{n+k-i}+B_{i/m}C_{i+n-k}-B_{i/m}C_{k-i-n}-B_{i/m}C_{i-n-k}-B_{i/m}C_{k+i-n})$$

$$\frac{\partial f_2^5}{\partial c_{2n}}=\frac{1}{4}\sum_{i=1}^{N}(B_{i/m}Q_{i+n+k}+B_{i/m}Q_{n-i-k}+B_{i/m}Q_{n+k-i}+B_{i/m}Q_{i+n-k}-B_{i/m}Q_{k-i-n}-B_{i/m}Q_{i-n-k}-B_{i/m}Q_{k+i-n})$$
$$+\frac{1}{4}\sum_{j=1}^{N}(C_{j/m}P_{j-n-k}+C_{j/m}P_{j+k-n}+C_{j/m}P_{n+k-j}+C_{j/m}P_{n-k-j}-C_{j/m}P_{k-n-j}-C_{j/m}P_{n+k+j}$$
$$-C_{j/m}P_{n+j-k})$$

$$\frac{\partial f_2^6}{\partial c_{2n}}=a_{20}(C_{n-k}+C_{n+k}-C_{k-n})+\frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}B_{k+i-n}+c_{2i/m}B_{i-n-k}+c_{2i/m}B_{n-i-k}+c_{2i/m}B_{n+k-i}-c_{2i/m}B_{i+n-k}$$
$$-c_{2i/m}B_{k-i-n}-c_{2i/m}B_{i+n+k})+\frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}C_{i+n+k}+b_{2i/m}C_{n-i-k}+b_{2i/m}C_{n+k-i}+b_{2i/m}C_{i+n-k}-b_{2i/m}C_{k-i-n}$$
$$-b_{2i/m}C_{i-n-k}-b_{2i/m}C_{k+i-n})$$

$$\frac{\partial f_2^7}{\partial c_{2n}}=\delta_n^k k\Omega$$

$$\frac{\partial f_2^9}{\partial c_{2n}}=\frac{1}{2}\sum_{i=1}^{N}(P_{i/m}Q_{i+n+k}+P_{i/m}Q_{n-i-k}+P_{i/m}Q_{n+k-i}+P_{i/m}Q_{i+n-k}-P_{i/m}Q_{k-i-n}-P_{i/m}Q_{i-n-k}-P_{i/m}Q_{k+i-n})$$

$$\frac{\partial f_2^{10}}{\partial c_{2n}}=a_{20}(Q_{n-k}+Q_{n+k}-Q_{k-n})+\frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}Q_{i+n+k}+b_{2i/m}Q_{n-i-k}+b_{2i/m}Q_{n+k-i}+b_{2i/m}Q_{i+n-k}-$$
$$b_{2i/m}Q_{k-i-n}-b_{2i/m}Q_{i-n-k}-b_{2i/m}Q_{k+i-n})\frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}P_{k+i-n}+c_{2i/m}P_{i-n-k}+c_{2i/m}P_{n-i-k}+c_{2i/m}P_{n+k-i}-$$
$$c_{2i/m}P_{i+n-k}-c_{2i/m}P_{k-i-n}-c_{2i/m}P_{i+n+k})$$

$$\frac{\partial f_2^{12}}{\partial c_{2n}} = 3a_{20}(c_{2(n-k)} + c_{2(n+k)} - c_{2(k-n)}) + \frac{3}{2}\sum_{i=1}^{N}(b_{2i/m}c_{2(i+n+k)} + b_{2i/m}c_{2(n-i-k)} + b_{2i/m}c_{2(n+k-i)} + b_{2i/m}c_{2(i+n-k)}$$
$$- b_{2i/m}c_{2(k-i-n)} - b_{2i/m}c_{2(i-n-k)} - b_{2i/m}c_{2(k+i-n)})$$

$$\frac{\partial f_2^{14}}{\partial c_{2n}} = \delta_n^k k\Omega$$

B.3 Mathematical expression for the derivative of the sine term.

$$\frac{\partial f_3^{3}}{\partial a_{20}} = \sum_{i=1}^{N}(B_{i/m}C_{i+k} + B_{i/m}C_{k-i} - B_{i/m}C_{i-k})$$

$$\frac{\partial f_3^{5}}{\partial a_{20}} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}Q_{i+k} + B_{i/m}Q_{k-i} - B_{i/m}Q_{i-k}) + \frac{1}{2}\sum_{i=1}^{N}(C_{i/m}P_{i-k} + C_{i/m}P_{k-i} - C_{i/m}P_{i+k})$$

$$\frac{\partial f_3^{6}}{\partial a_{20}} = 2a_{20}C_{k/m} + \sum_{i=1}^{N}(b_{2i/m}C_{i+k} + b_{2i/m}C_{k-i} - b_{2i/m}C_{i-k}) + \sum_{i=1}^{N}(c_{2i/m}B_{k-i} + c_{2i/m}B_{i-k} - c_{2i/m}B_{i+k})$$

$$\frac{\partial f_3^{9}}{\partial a_{20}} = \sum_{i=1}^{N}(P_{i/m}Q_{i+k} + P_{i/m}Q_{k-i} - P_{i/m}Q_{i-k})$$

$$\frac{\partial f_3^{10}}{\partial a_{20}} = 2a_{20}Q_{k/m} + \sum_{i=1}^{N}(b_{2i/m}Q_{i+k} + b_{2i/m}Q_{k-i} - b_{2i/m}Q_{i-k}) + \sum_{i=1}^{N}(c_{2i/m}P_{k-i} + c_{2i/m}P_{i-k} - c_{2i/m}P_{i+k})$$

$$\frac{\partial f_3^{12}}{\partial a_{20}} = 6a_{20}c_{2k/m} + 3\sum_{i=1}^{N}(b_{2i/m}c_{2(i+k)} + b_{2i/m}c_{2(k-i)} - b_{2i/m}c_{2(i-k)})$$

$$\frac{\partial f_3^{2}}{\partial Q_n} = \frac{1}{4}\sum_{j=1}^{N}(B_{j/m}B_{k-n-j} + B_{j/m}B_{n+k+j} + B_{j/m}B_{j+k-n} + B_{j/m}B_{n+k-j} - B_{j/m}B_{n-k-j} - B_{j/m}B_{j-n-k}$$
$$- B_{j/m}B_{n+j-k}) + \frac{1}{4}\sum_{j=1}^{N}(C_{j/m}C_{n+j-k} + C_{j/m}C_{j+k-n} + C_{j/m}C_{n+k-j} - C_{j/m}C_{n-k-j} - C_{j/m}C_{k-n-j}$$
$$- C_{j/m}C_{n+k+j} - C_{j/m}C_{j-n-k})$$

$$\frac{\partial f_3^{4}}{\partial Q_n} = \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}Q_{j+k-n} + C_{j/m}Q_{n+k-j} + C_{j/m}Q_{n+j-k} - C_{j/m}Q_{n-k-j} - C_{j/m}Q_{k-n-j} - C_{j/m}Q_{n+k+j}$$
$$- C_{j/m}Q_{j-n-k}) + \frac{1}{2}\sum_{j=1}^{N}(B_{j/m}P_{j+k-n} + B_{j/m}P_{n+k-j} + B_{j/m}P_{k-n-j} + B_{j/m}P_{n+k+j} - B_{j/m}P_{n-k-j}$$
$$- B_{j/m}P_{n+j-k} - B_{j/m}P_{j-n-k})$$

$$\frac{\partial f_3^{5}}{\partial Q_n} = \frac{1}{2}a_{20}(B_{n-k} + B_{k-n} - B_{n+k}) + \frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}B_{j+k-n} + b_{2j/m}B_{n+k-j} + b_{2j/m}B_{k-n-j} + b_{2j/m}B_{n+k+j}$$
$$- b_{2j/m}B_{n-k-j} - b_{2j/m}B_{n+j-k} - b_{2j/m}B_{j-n-k}) + \frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}C_{k+i-n} + c_{2i/m}C_{i+n-k} + c_{2i/m}C_{n+k-i} -$$
$$c_{2i/m}C_{k-i-n} - c_{2i/m}C_{i-n-k} - c_{2i/m}C_{i+n+k} - c_{2i/m}C_{n-i-k})$$

$$\frac{\partial f_3^8}{\partial Q_n} = \frac{3}{4}\sum_{i=1}^{N}(Q_{i/m}Q_{k+i-n} + Q_{i/m}Q_{i+n-k} + Q_{i/m}Q_{n+k-i} - Q_{i/m}Q_{i-n-k} - Q_{i/m}Q_{i+n+k} - Q_{i/m}Q_{n-i-k} - Q_{i/m}Q_{k-i-n})$$

$$+\frac{3}{4}\sum_{j=1}^{N}(P_{j/m}P_{n+j-k} + P_{j/m}P_{j+k-n} + P_{j/m}P_{n-k-j} + P_{j/m}P_{k-n-j} - P_{j/m}P_{j-n-k} - P_{j/m}P_{n+k+j} - P_{j/m}P_{n+k-j})$$

$$\frac{\partial f_3^9}{\partial Q_n} = a_{20}(P_{n-k} + P_{k-n} - P_{n+k}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}P_{j+k-n} + b_{2j/m}P_{n+k-j} + b_{2j/m}P_{k-n-j} + b_{2j/m}P_{n+k+j} -$$

$$b_{2j/m}P_{n+j-k} - b_{2j/m}P_{j-n-k} - b_{2j/m}P_{n-k-j}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}Q_{k+i-n} + c_{2i/m}Q_{i+n-k} + c_{2i/m}Q_{n+k-i} - c_{2i/m}Q_{k-i-n}$$

$$- c_{2i/m}Q_{i-n-k} - c_{2i/m}Q_{i+n+k} - c_{2i/m}Q_{n-i-k})$$

$$\frac{\partial f_3^{10}}{\partial Q_n} = a_{20}(b_{2(n-k)} + b_{2(k-n)} - b_{2(n+k)}) + \frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}b_{2(j+k-n)} + b_{2j/m}b_{2(n+k-j)} + b_{2j/m}b_{2(k-n-j)} +$$

$$b_{2j/m}b_{2(n+k+j)} - b_{2j/m}b_{2(n+j-k)} - b_{2j/m}b_{2(j-n-k)} - b_{2j/m}b_{2(n-k-j)}) + \frac{1}{4}\sum_{j=1}^{N}(c_{2j/m}c_{2(n+j-k)} +$$

$$c_{2j/m}c_{2(j+k-n)} + c_{2j/m}c_{2(n+k-j)} - c_{2j/m}c_{2(k-n-j)} - c_{2j/m}c_{2(n+k+j)} - c_{2j/m}c_{2(j-n-k)} - c_{2j/m}c_{2(n-k-j)})$$

$$- \delta_n^k a_{20}^2 k\Omega$$

$$\frac{\partial f_3^2}{\partial P_n} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}C_{i+n+k} + B_{i/m}C_{k+i-n} + B_{i/m}C_{k-i-n} + B_{i/m}C_{n+k-i} - B_{i/m}C_{i+n-k} - B_{i/m}C_{i-n-k} - B_{i/m}C_{n-i-k})$$

$$\frac{\partial f_3^4}{\partial P_n} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}Q_{k+i-n} + B_{i/m}Q_{n-i-k} + B_{i/m}Q_{i+n-k} + B_{i/m}Q_{k-i-n} - B_{i/m}Q_{i-n-k} - B_{i/m}Q_{i+n+k} - B_{i/m}Q_{n+k-i})$$

$$+\frac{1}{2}\sum_{j=1}^{N}(C_{j/m}P_{n+j-k} + C_{j/m}P_{j-n-k} + C_{j/m}P_{n+k-j} + C_{j/m}P_{k-n-j} - C_{j/m}P_{j+k-n} - C_{j/m}P_{n+k+j} - C_{j/m}P_{n-k-j})$$

$$\frac{\partial f_3^5}{\partial P_n} = \frac{1}{2}a_{20}(C_{n+k} + C_{k-n} - C_{n-k}) + \frac{1}{4}\sum_{i=1}^{N}b_{2i/m}C_{i+n+k} + b_{2i/m}C_{n+k-i} + b_{2i/m}C_{k+i-n} - b_{2i/m}C_{i+n-k} +$$

$$b_{2i/m}C_{k-i-n} - b_{2i/m}C_{i-n-k} - b_{2i/m}C_{n-i-k}) + \frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}B_{i-n-k} + c_{2i/m}B_{n+k-i} + c_{2i/m}B_{i+n-k} + c_{2i/m}B_{k-i-n}$$

$$- c_{2i/m}B_{i+n+k} - c_{2i/m}B_{n-i-k} - c_{2i/m}B_{k+i-n})$$

$$\frac{\partial f_3^8}{\partial P_n} = \frac{3}{2}\sum_{i=1}^{N}(P_{i/m}Q_{i+n+k} + P_{i/m}Q_{n+k-i} + P_{i/m}Q_{k-i-n} + P_{i/m}Q_{k+i-n} - P_{i/m}Q_{n-i-k} - P_{i/m}Q_{i-n-k} - P_{i/m}Q_{i+n-k})$$

$$\frac{\partial f_3^9}{\partial P_n} = a_{20}(Q_{k-n} + Q_{n+k} - Q_{n-k}) + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}Q_{n-i-k} + b_{2i/m}Q_{k+i-n} + b_{2i/m}Q_{i+n-k} + b_{2i/m}Q_{k-i-n}$$

$$- b_{2i/m}Q_{i+n+k} - b_{2i/m}Q_{n+k-i} - b_{2i/m}Q_{i-n-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}P_{i-n-k} + c_{2i/m}P_{n+k-i} + c_{2i/m}P_{i+n-k} + c_{2i/m}P_{k-i-n}$$

$$- c_{2i/m}P_{i+n+k} - c_{2i/m}P_{n-i-k} - c_{2i/m}P_{k+i-n})]$$

$$\frac{\partial f_3^{10}}{\partial P_n} = a_{20}(c_{2(n+k)} + c_{2(k-n)} - c_{2(n-k)}) + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}c_{2(i+n+k)} + b_{2i/m}c_{2(k+i-n)} + b_{2i/m}c_{2(n+k-i)} + b_{2i/m}c_{2(k-i-n)}$$
$$- b_{2i/m}c_{2(i-n-k)} - b_{2i/m}c_{2(n-i-k)} - b_{2i/m}c_{2(i+n-k)})$$

$$\frac{\partial f_3^{1}}{\partial C_n} = \frac{3}{4}\sum_{j=1}^{N}(C_{j/m}C_{n+j-k} + C_{j/m}C_{j+k-n} + C_{j/m}C_{n+k-j} - C_{j/m}C_{n-k-j} - C_{j/m}C_{k-n-j} - C_{j/m}C_{n+k+j}$$
$$- C_{j/m}C_{j-n-k}) + \sum_{j=1}^{N}(B_{j/m}B_{k-n-j} + B_{j/m}B_{j+k-n} + B_{j/m}B_{n+j-k} + B_{j/m}B_{n-k-j} - B_{j/m}B_{n+k-j} - B_{j/m}B_{n+k+j}$$
$$- B_{j/m}B_{j-n-k})$$

$$\frac{\partial f_3^{2}}{\partial C_n} = \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}Q_{j+k-n} + C_{j/m}Q_{n+k-j} + C_{j/m}Q_{n+j-k} - C_{j/m}Q_{n-k-j} - C_{j/m}Q_{k-n-j} - C_{j/m}Q_{n+k+j}$$
$$- C_{j/m}Q_{j-n-k}) + \sum_{j=1}^{N}(B_{j/m}P_{j+k-n} + B_{j/m}P_{k-n-j} + B_{j/m}P_{n-k-j} + B_{j/m}P_{n+j-k} - B_{j/m}P_{n+k+j} - B_{j/m}P_{n+k-j}$$
$$- B_{j/m}P_{j-n-k})$$

$$\frac{\partial f_3^{3}}{\partial C_n} = a_{20}(B_{n-k} + B_{k-n} - B_{n+k}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}B_{j+k-n} + b_{2j/m}B_{n+k-j} + b_{2j/m}B_{k-n-j} + b_{2j/m}B_{n+k+j} -$$
$$b_{2j/m}B_{n-k-j} - b_{2j/m}B_{n+j-k} - b_{2j/m}B_{j-n-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}C_{k+i-n} + c_{2i/m}C_{i+n-k} + c_{2i/m}C_{n+k-i} -$$
$$c_{2i/m}C_{k-i-n} - c_{2i/m}C_{i-n-k} - c_{2i/m}C_{i+n+k} - c_{2i/m}C_{n-i-k})$$

$$\frac{\partial f_3^{4}}{\partial C_n} = \frac{1}{4}[\sum_{j=1}^{N}(P_{j/m}P_{n+j-k} + P_{j/m}P_{j+k-n} + P_{j/m}P_{n-k-j} + P_{j/m}P_{k-n-j} - P_{j/m}P_{j-n-k} - P_{j/m}P_{n+k+j} - P_{j/m}P_{n+k-j})$$
$$+ \frac{1}{4}\sum_{i=1}^{N}(Q_{i/m}Q_{k+i-n} + Q_{i/m}Q_{i+n-k} + Q_{i/m}Q_{n+k-i} - Q_{i/m}Q_{i-n-k} - Q_{i/m}Q_{i+n+k} - Q_{i/m}Q_{n-i-k} - Q_{i/m}Q_{k-i-n})$$

$$\frac{\partial f_3^{5}}{\partial C_n} = \frac{1}{2}a_{20}(P_{n-k} + P_{k-n} - P_{n+k}) + \frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}P_{j+k-n} + b_{2j/m}P_{n+j-k} + b_{2j/m}P_{n-k-j} + b_{2j/m}P_{k-n-j}$$
$$- b_{2j/m}P_{j-n-k} - b_{2j/m}P_{n+k-j} - b_{2j/m}P_{n+k+j}) + \frac{1}{4}\sum_{i=1}^{N}(c_{2i/m}Q_{k+i-n} + c_{2i/m}Q_{n+k-i} + c_{2i/m}Q_{i+n-k}$$
$$- c_{2i/m}Q_{n-i-k} - c_{2i/m}Q_{k-i-n} - c_{2i/m}Q_{i-n-k} - c_{2i/m}Q_{i+n+k})$$

$$\frac{\partial f_3^{6}}{\partial C_n} = a_{20}(b_{2(n-k)} + b_{2(k-n)} - b_{2(n+k)}) + \frac{1}{4}\sum_{j=1}^{N}(b_{2j/m}b_{2(j+k-n)} + b_{2j/m}b_{2(n+k-j)} + b_{2j/m}b_{2(k-n-j)} +$$
$$b_{2j/m}b_{2(n+k+j)} - b_{2j/m}b_{2(n+j-k)} - b_{2j/m}b_{2(j-n-k)} - b_{2j/m}b_{2(n-k-j)}) + \frac{1}{4}\sum_{j=1}^{N}(c_{2j/m}c_{2(n+j-k)} +$$
$$c_{2j/m}c_{2(j+k-n)} + c_{2j/m}c_{2(n+k-j)} - c_{2j/m}c_{2(k-n-j)} - c_{2j/m}c_{2(n+k+j)} - c_{2j/m}c_{2(j-n-k)} - c_{2j/m}c_{2(n-k-j)})$$
$$- a_{20}^2(k\Omega)\delta_n^k$$

$$\frac{\partial f_3^3}{\partial b_{2n}} = \frac{1}{2}\sum_{j=1}^{N}(B_{j/m}C_{n+j-k} + B_{j/m}C_{n+k-j} + B_{j/m}C_{k-n-j} + B_{j/m}C_{j-n-k} - B_{j/m}C_{n-k-j} - B_{j/m}C_{j+k-n} - B_{j/m}C_{n+k+j})$$

$$\frac{\partial f_3^5}{\partial b_{2n}} = \frac{1}{4}\sum_{j=1}^{N}(B_{j/m}Q_{k-n-j} + B_{j/m}Q_{j-n-k} + B_{j/m}Q_{n+j-k} + B_{j/m}Q_{n+k-j} - B_{j/m}Q_{n-k-j} - B_{j/m}Q_{n+k+j} - B_{j/m}Q_{j+k-n}) + \frac{1}{4}\sum_{j=1}^{N}(C_{j/m}P_{n+j-k} + C_{j/m}P_{j-n-k} + C_{j/m}P_{n+k-j} + C_{j/m}P_{k-n-j} - C_{j/m}P_{j+k-n} - C_{j/m}P_{n+k+j} - C_{j/m}P_{n-k-j})$$

$$\frac{\partial f_3^6}{\partial b_{2n}} = a_{20}(C_{n+k} + C_{k-n} - C_{n-k}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}C_{n-k-j} + b_{2j/m}C_{j+k-n} + b_{2j/m}C_{n+j-k} + b_{2j/m}C_{k-n-j} - b_{2j/m}C_{n+k+j} - b_{2j/m}C_{n+k-j} - b_{2j/m}C_{j-n-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}B_{i-n-k} + c_{2i/m}B_{i+n-k} + c_{2i/m}B_{n+k-i} + c_{2i/m}B_{k-i-n} - c_{2i/m}B_{i+n+k} - c_{2i/m}B_{n-i-k} - c_{2i/m}B_{k+i-n})$$

$$\frac{\partial f_3^7}{\partial b_{2n}} = -(k\Omega)\delta_n^k$$

$$\frac{\partial f_3^9}{\partial b_{2n}} = \frac{1}{2}\sum_{j=1}^{N}(P_{j/m}Q_{n+j-k} + P_{j/m}Q_{k-n-j} + P_{j/m}Q_{j-n-k} + P_{j/m}Q_{n+k-j} - P_{j/m}Q_{n+k+j} - P_{j/m}Q_{j+k-n} - P_{j/m}Q_{n-k-j})$$

$$\frac{\partial f_3^{10}}{\partial b_{2n}} = a_{20}(Q_{k-n} + Q_{n+k} - Q_{n-k}) + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}Q_{n-i-k} + b_{2i/m}Q_{k+i-n} + b_{2i/m}Q_{i+n-k} + b_{2i/m}Q_{k-i-n} - b_{2i/m}Q_{i+n+k} - b_{2i/m}Q_{n+k-i} - b_{2i/m}Q_{i-n-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}P_{i-n-k} + c_{2i/m}P_{n+k-i} + c_{2i/m}P_{i+n-k} + c_{2i/m}P_{k-i-n} - c_{2i/m}P_{i+n+k} - c_{2i/m}P_{n-i-k} - c_{2i/m}P_{k+i-n})$$

$$\frac{\partial f_3^{12}}{\partial b_{2n}} = 3a_{20}(c_{2(n+k)} + c_{2(k-n)} - c_{2(n-k)}) + \frac{3}{2}\sum_{i=1}^{N}(b_{2i/m}c_{2(i+n+k)} + b_{2i/m}c_{2(k+i-n)} + b_{2i/m}c_{2(n+k-i)} + b_{2i/m}c_{2(k-i-n)} - b_{2i/m}c_{2(i-n-k)} - b_{2i/m}c_{2(n-i-k)} - b_{2i/m}c_{2(i+n-k)})$$

$$\frac{\partial f_3^{14}}{\partial b_{2n}} = -k\Omega\delta_n^k$$

$$\frac{\partial f_3^1}{\partial B_n} = \frac{3}{2}\sum_{i=1}^{N}(B_{i/m}C_{i+n+k} + B_{i/m}C_{k+i-n} + B_{i/m}C_{k-i-n} + B_{i/m}C_{n+k-i} - B_{i/m}C_{i+n-k} - B_{i/m}C_{i-n-k} - B_{i/m}C_{n-i-k})$$

$$\frac{\partial f_3^2}{\partial B_n} = \frac{1}{4}\sum_{j=1}^{N}(B_{j/m}Q_{k-n-j} + B_{j/m}Q_{j-n-k} + B_{j/m}Q_{n+j-k} + B_{j/m}Q_{n+k-j} - B_{j/m}Q_{n-k-j} - B_{j/m}Q_{n+k+j} - B_{j/m}Q_{j+k-n}) + \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}P_{n+j-k} + C_{j/m}P_{j-n-k} + C_{j/m}P_{n+k-j} + C_{j/m}P_{k-n-j} - C_{j/m}P_{j+k-n}$$

$$-C_{j/m}P_{n+k+j}-C_{j/m}P_{n-k-j})+\frac{1}{4}\sum_{i=1}^{N}(B_{i/m}Q_{k+i-n}+B_{i/m}Q_{n-i-k}+B_{i/m}Q_{i+n-k}+B_{i/m}Q_{k-i-n}$$

$$-B_{i/m}Q_{i-n-k}-B_{i/m}Q_{i+n+k}-B_{i/m}Q_{n+k-i})$$

$$\frac{\partial f_3^3}{\partial B_n}=a_{20}(C_{n+k}+C_{k-n}-C_{n-k})+\frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}C_{n-k-j}+b_{2j/m}C_{j+k-n}+b_{2j/m}C_{n+j-k}+b_{2j/m}C_{k-n-j}$$

$$-b_{2j/m}C_{n+k+j}-b_{2j/m}C_{n+k-j}-b_{2j/m}C_{j-n-k})+\frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}B_{i-n-k}+c_{2i/m}B_{i+n-k}+c_{2i/m}B_{n+k-i}$$

$$+c_{2i/m}B_{k-i-n}-c_{2i/m}B_{i+n+k}-c_{2i/m}B_{n-i-k}-c_{2i/m}B_{k+i-n})$$

$$\frac{\partial f_3^4}{\partial B_n}=\sum_{j=1}^{N}(P_{j/m}Q_{n+j-k}+P_{j/m}Q_{k-n-j}+P_{j/m}Q_{j-n-k}+P_{j/m}Q_{n+k-j}-P_{j/m}Q_{n+k+j}-P_{j/m}Q_{j+k-n}-P_{j/m}Q_{n-k-j})$$

$$\frac{\partial f_3^5}{\partial B_n}=\frac{1}{2}a_{20}(Q_{k-n}+Q_{n+k}-Q_{n-k})+\frac{1}{4}\sum_{i=1}^{N}(b_{2i/m}Q_{n-i-k}+b_{2i/m}Q_{k+i-n}+b_{2i/m}Q_{i+n-k}+b_{2i/m}Q_{k-i-n}-$$

$$b_{2i/m}Q_{i+n+k}-b_{2i/m}Q_{n+k-i}-b_{2i/m}Q_{i-n-k})]+\frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}P_{i-n-k}+c_{2i/m}P_{i+n-k}+c_{2i/m}P_{n+k-i}+c_{2i/m}P_{k-i-n}$$

$$-c_{2i/m}P_{i+n+k}-c_{2i/m}P_{n-i-k}-c_{2i/m}P_{k+i-n})$$

$$\frac{\partial f_3^6}{\partial B_n}=a_{20}(c_{2(n+k)}+c_{2(k-n)}-c_{2(n-k)})+\frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}c_{2(i+n+k)}+b_{2i/m}c_{2(k+i-n)}+b_{2i/m}c_{2(n+k-i)}+$$

$$b_{2i/m}c_{2(k-i-n)}-b_{2i/m}c_{2(i-n-k)}-b_{2i/m}c_{2(n-i-k)}-b_{2i/m}c_{2(i+n-k)})$$

$$\frac{\partial f_3^3}{\partial c_{2n}}=\sum_{j=1}^{N}(B_jB_{k-n-j}+B_jB_{j+k-n}+B_jB_{n+j-k}+B_jB_{n-k-j}-B_jB_{n+k-j}-B_jB_{n+k+j}-B_jB_{j-n-k})$$

$$+\sum_{j=1}^{N}(C_jC_{n+j-k}+C_jC_{j+k-n}+C_jC_{n+k-j}-C_jC_{n-k-j}-C_jC_{k-n-j}-C_jC_{n+k+j}-C_jC_{j-n-k})$$

$$\frac{\partial f_3^5}{\partial c_{2n}}=\frac{1}{4}\sum_{j=1}^{N}(B_{j/m}P_{k-n-j}+B_{j/m}P_{j+k-n}+B_{j/m}P_{n+j-k}+B_{j/m}P_{n-k-j}-B_{j/m}P_{n+k-j}-B_{j/m}P_{n+k+j}$$

$$-B_{j/m}P_{j-n-k})+\frac{1}{4}\sum_{j=1}^{N}(C_{j/m}Q_{j+k-n}+C_{j/m}Q_{n+k-j}+C_{j/m}Q_{n+j-k}-C_{j/m}Q_{n-k-j}-C_{j/m}Q_{k-n-j}$$

$$-C_{j/m}Q_{n+k+j}-C_{j/m}Q_{j-n-k})$$

$$\frac{\partial f_3^6}{\partial c_{2n}}=a_{20}(B_{n-k}+B_{k-n}-B_{n+k})+\frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}B_{n-i-k}+b_{2i/m}B_{k+i-n}+b_{2i/m}B_{i+n-k}+b_{2i/m}B_{k-i-n}$$

$$-b_{2i/m}B_{i+n+k}-b_{2i/m}B_{i-n-k}-b_{2i/m}B_{n+k-i})+\frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}C_{k+i-n}+c_{2i/m}C_{i+n-k}+c_{2i/m}C_{n+k-i}$$

$$-c_{2i/m}C_{k-i-n}-c_{2i/m}C_{i-n-k}-c_{2i/m}C_{i+n+k}-c_{2i/m}C_{n-i-k})$$

$$\frac{\partial f_3^9}{\partial c_{2n}} = \frac{1}{4}\sum_{j=1}^{N}(P_{j/m}P_{n+j-k} + P_{j/m}P_{j+k-n} + P_{j/m}P_{n-k-j} + P_{j/m}P_{k-n-j} - P_{j/m}P_{j-n-k} - P_{j/m}P_{n+k+j} - P_{j/m}P_{n+k-j}) +$$

$$\frac{1}{4}\sum_{i=1}^{N}(Q_{i/m}Q_{k+i-n} + Q_{i/m}Q_{i+n-k} + Q_{i/m}Q_{n+k-i} - Q_{i/m}Q_{i-n-k} - Q_{i/m}Q_{i+n+k} - Q_{i/m}Q_{n-i-k} - Q_{i/m}Q_{k-i-n})$$

$$\frac{\partial f_3^{10}}{\partial c_{2n}} = a_{20}(P_{n-k} + P_{k-n} - P_{n+k}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}P_{j+k-n} + b_{2j/m}P_{n+j-k} + b_{2j/m}P_{n-k-j} + b_{2j/m}P_{k-n-j} -$$

$$b_{2j/m}P_{j-n-k} - b_{2j/m}P_{n+k-j} - b_{2j/m}P_{n+k+j}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}Q_{k+i-n} + c_{2i/m}Q_{n+k-i} + c_{2i/m}Q_{i+n-k}$$

$$- c_{2i/m}Q_{n-i-k} - c_{2i/m}Q_{k-i-n} - c_{2i/m}Q_{i-n-k} - c_{2i/m}Q_{i+n+k})$$

$$\frac{\partial f_3^{12}}{\partial c_{2n}} = 3a_{20}(b_{2(n-k)} + b_{2(k-n)} - b_{2(n+k)}) + \frac{3}{4}\sum_{j=1}^{N}(b_{2j/m}b_{2(j+k-n)} + b_{2j/m}b_{2(k-n-j)} + b_{2j/m}b_{2(n-k-j)}$$

$$+ b_{2j/m}b_{2(n+j-k)} - b_{2j/m}b_{2(n+k+j)} - b_{2j/m}b_{2(n+k-j)} - b_{2j/m}b_{2(j-n-k)}) + \frac{3}{4}\sum_{j=1}^{N}(c_{2j/m}c_{2(n+j-k)} +$$

$$c_{2j/m}c_{2(j+k-n)} + c_{2j/m}c_{2(n+k-j)} - c_{2j/m}c_{2(k-n-j)} - c_{2j/m}c_{2(n+k+j)} - c_{2j/m}c_{2(j-n-k)} - c_{2j/m}c_{2(n-k-j)})$$

$$+ 3a_{20}^2\delta_n^k$$

$$\frac{\partial f_3^{13}}{\partial c_{2n}} = \delta_n^k$$

$$\frac{\partial f_3^{15}}{\partial c_{2n}} = \delta_n^k$$

B.4 Additional terms for the derivative of nonlinear cable structure

$$\frac{\partial f_{11}^{16}}{\partial a_{10}} = 3a_{10}^2 + \frac{3}{2}\sum_{i=1}^{N}\left(b_{1i/m}^2 + c_{1i/m}^2\right)$$

$$\frac{\partial f_{11}^{16}}{\partial b_{1n}} = 3a_{10}c_{1n/m} + \frac{3}{2}\sum_{i=1}^{N}(b_{1i}c_{1(n-i)} + b_{1i}c_{1(n+i)} - b_{1i}c_{1(i-n)})$$

$$\frac{\partial f_{11}^{16}}{\partial c_{1n}} = 3a_{10}b_{1n/m} + \frac{3}{4}\sum_{i=1}^{N}(c_{1i}c_{1(n+i)} + c_{1i}c_{1(i-n)} - c_{1i}c_{1(n-i)}) + \frac{3}{4}\sum_{i=1}^{N}(b_{1i}b_{1(n+i)} + b_{1i}b_{1(i-n)} + b_{1i}b_{1(n-i)})$$

$$\frac{\partial f_{21}^{16}}{\partial a_{10}} = 6a_{10}b_{1r/m} + \frac{3}{2}\sum_{i=1}^{N}(b_{1i}b_{1(r+i)} + b_{1i}b_{1(i-r)} + b_{1i}b_{1(r-i)}) + \frac{3}{2}\sum_{i=1}^{N}(c_{1i}c_{1(r+i)} + c_{1i}c_{1(i-r)} - c_{1i}c_{1(r-i)})$$

$$\frac{\partial f_{21}^{16}}{\partial b_{1n}} = 3a_{10}(b_{1(n-k)} + b_{1(n+k)} + b_{1(k-n)}) + \frac{3}{4}\sum_{j=1}^{N}(b_{1j/m}b_{1(j-n-k)} + b_{1j/m}b_{1(j+k-n)} + b_{1j/m}b_{1(n+k-j)} + b_{1j/m}b_{1(n-k-j)}$$

$$+ b_{1j/m}b_{1(k-n-j)} + b_{1j/m}b_{1(n+k+j)} + b_{1j/m}b_{1(n+j-k)}) + \frac{3}{4}\sum_{j=1}^{N}(c_{1j/m}c_{1(j-n-k)} + c_{1j/m}c_{1(j+k-n)} - c_{1j/m}c_{1(n+k-j)}$$

$$- c_{1j/m}c_{1(n-k-j)} - c_{1j/m}c_{1(k-n-j)} + c_{1j/m}c_{1(n+k+j)} + c_{1j/m}c_{1(n+j-k)}) + \delta_n^k 3a_{10}^2$$

$$\frac{\partial f_{21}^{16}}{\partial c_{1n}} = 3a_{10}(c_{1(n-k)} + c_{1(n+k)} - c_{1(k-n)}) + \frac{3}{2}\sum_{i=1}^{N}(b_{1i/m}c_{1(i+n+k)} + b_{1i/m}c_{1(n-i-k)} + b_{1i/m}c_{1(n+k-i)} + b_{1i/m}c_{1(i+n-k)}$$

$$- b_{1i/m}c_{2(k-i-n)} - b_{1i/m}c_{1(i-n-k)} - b_{1i/m}c_{1(k+i-n)})$$

$$\frac{\partial f_{31}^{16}}{\partial a_{10}} = 6a_{10}c_{1k/m} + 3\sum_{i=1}^{N}(b_{1i/m}c_{1(i+k)} + b_{1i/m}c_{1(k-i)} - b_{1i/m}c_{1(i-k)})$$

$$\frac{\partial f_{31}^{16}}{\partial b_{1n}} = 3a_{10}(c_{1(n+k)} + c_{1(k-n)} - c_{1(n-k)}) + \frac{3}{2}\sum_{i=1}^{N}(b_{1i/m}c_{1(i+n+k)} + b_{1i/m}c_{1(k+i-n)} + b_{1i/m}c_{1(n+k-i)}$$

$$+ b_{1i/m}c_{1(k-i-n)} - b_{1i/m}c_{1(i-n-k)} - b_{1i/m}c_{1(n-i-k)} - b_{1i/m}c_{1(i+n-k)})$$

$$\frac{\partial f_{31}^{16}}{\partial c_{1n}} = 3a_{10}(b_{1(n-k)} + b_{1(k-n)} - b_{1(n+k)}) + \frac{3}{4}\sum_{j=1}^{N}(b_{1j/m}b_{1(j+k-n)} + b_{1j/m}b_{1(k-n-j)} + b_{1j/m}b_{1(n-k-j)}$$

$$+ b_{1j/m}b_{1(n+j-k)} - b_{1j/m}b_{1(n+k+j)} - b_{1j/m}b_{1(n+k-j)} - b_{1j/m}b_{1(j-n-k)}) + \frac{3}{4}\sum_{j=1}^{N}(c_{1j/m}c_{1(n+j-k)} + c_{1j/m}c_{1(j+k-n)}$$

$$+ c_{1j/m}c_{1(n+k-j)} - c_{1j/m}c_{1(k-n-j)} - c_{1j/m}c_{1(n+k+j)} - c_{1j/m}c_{1(j-n-k)} - c_{1j/m}c_{1(n-k-j)}) + 3a_{10}^{2}\delta_{n}^{k}$$

$$\frac{\partial f_{12}^{16}}{\partial a_{20}} = \frac{\partial f_{1}^{12}}{\partial a_{20}}, \frac{\partial f_{12}^{16}}{\partial b_{2n}} = \frac{\partial f_{1}^{12}}{\partial b_{2n}}, \frac{\partial f_{12}^{16}}{\partial c_{2n}} = \frac{\partial f_{1}^{12}}{\partial c_{2n}},$$

$$\frac{\partial f_{22}^{16}}{\partial a_{20}} = \frac{\partial f_{2}^{12}}{\partial a_{20}}, \frac{\partial f_{22}^{16}}{\partial b_{2n}} = \frac{\partial f_{2}^{12}}{\partial b_{2n}}, \frac{\partial f_{22}^{16}}{\partial c_{2n}} = \frac{\partial f_{2}^{12}}{\partial c_{2n}},$$

$$\frac{\partial f_{32}^{16}}{\partial a_{20}} = \frac{\partial f_{3}^{12}}{\partial a_{20}}, \frac{\partial f_{32}^{16}}{\partial b_{2n}} = \frac{\partial f_{3}^{12}}{\partial b_{2n}}, \frac{\partial f_{32}^{16}}{\partial c_{2n}} = \frac{\partial f_{3}^{12}}{\partial c_{2n}},$$

APPENDIX C

C.1 Mathematical expression for the derivative of time derivative of constant.

$$\frac{\partial f_1^2}{\partial \dot{a}_{10}} = \frac{1}{2}\sum_{i=1}^{N}(B_{i/m}^2 + C_{i/m}^2) + \dot{a}_{20}^2,\ \frac{\partial f_1^4}{\partial \dot{a}_{10}} = \sum_{i=1}^{N}(B_{i/m}P_{i/m} + C_{i/m}Q_{i/m}) + 2\dot{a}_{20}\dot{a}_{10}$$

$$\frac{\partial f_1^5}{\partial \dot{a}_{10}} = \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}B_{i/m} + c_{2i/m}C_{i/m}) + a_{20}\dot{a}_{20},\ \frac{\partial f_1^8}{\partial \dot{a}_{10}} = \frac{3}{2}\sum_{i=1}^{N}(P_{i/m}^2 + Q_{i/m}^2) + 3\dot{a}_{10}^2$$

$$\frac{\partial f_1^9}{\partial \dot{a}_{10}} = \sum_{i=1}^{N}(b_{2i/m}P_{i/m} + c_{2i/m}Q_{i/m}) + 2a_{20}\dot{a}_{10},\ \frac{\partial f_1^{10}}{\partial \dot{a}_{10}} = a_{20}^2 + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}^2 + c_{2i/m}^2)$$

$$\frac{\partial f_1^1}{\partial \dot{a}_{20}} = 3\frac{\partial f_1^2}{\partial \dot{a}_{10}},\ \frac{\partial f_1^2}{\partial \dot{a}_{20}} = \frac{\partial f_1^4}{\partial \dot{a}_{10}},\ \frac{\partial f_1^3}{\partial \dot{a}_{20}} = 2\frac{\partial f_1^5}{\partial \dot{a}_{10}},\ \frac{\partial f_1^4}{\partial \dot{a}_{20}} = 3\frac{\partial f_1^8}{\partial \dot{a}_{10}},\ \frac{\partial f_1^5}{\partial \dot{a}_{20}} = 2\frac{\partial f_1^9}{\partial \dot{a}_{10}},\ \frac{\partial f_1^6}{\partial \dot{a}_{20}} = \frac{\partial f_1^{10}}{\partial \dot{a}_{10}}$$

$$\frac{\partial f_2^2}{\partial \dot{a}_{10}} = 2\dot{a}_{20}B_{k/m} + \frac{1}{2}\sum_{j=1}^{N}(B_{j-k}B_{j/m} + B_{k-j}B_{j/m} + B_{k+j}B_{j/m}) + \frac{1}{2}\sum_{j=1}^{N}(C_{j/m}C_{j+k} + C_{j/m}C_{j-k} - C_{j/m}C_{k-j})$$

$$\frac{\partial f_2^4}{\partial \dot{a}_{10}} = 2\dot{a}_{20}P_{k/m} + 2\dot{a}_{10}B_{k/m} + \sum_{j=1}^{N}(B_{j/m}P_{j+k} + B_{j/m}P_{j-k} + B_{j/m}P_{k-j}) + \sum_{j=1}^{N}(C_{j/m}Q_{j+k} + C_{j/m}Q_{j-k} - C_{j/m}Q_{k-j})$$

$$\frac{\partial f_2^5}{\partial \dot{a}_{10}} = a_{20}B_{k/m} + \dot{a}_{20}b_{2k/m}\frac{1}{2}\sum_{j=1}^{N}(c_{2j/m}C_{j+k} + c_{2j/m}C_{j-k} - c_{2j/m}C_{k-j}) + \frac{1}{2}\sum_{j=1}^{N}(b_{2j/m}B_{j+k} + b_{2j/m}B_{j-k} + b_{2j/m}B_{k-j})$$

$$\frac{\partial f_2^8}{\partial \dot{a}_{10}} = 6\dot{a}_{10}P_{k/m} + \frac{3}{2}\sum_{j=1}^{N}(P_{j/m}P_{j+k} + P_{j/m}P_{j-k} + P_{j/m}P_{k-j}) + \frac{3}{2}\sum_{j=1}^{N}(Q_{j/m}Q_{j+k} + Q_{j/m}Q_{j-k} - Q_{j/m}Q_{k-j})$$

$$\frac{\partial f_2^9}{\partial \dot{a}_{10}} = 2\dot{a}_{20}P_{k/m} + 2\dot{a}_{10}b_{2k/m} + \sum_{j=1}^{N}(b_{2j/m}P_{j+k} + b_{2j/m}P_{j-k} + b_{2j/m}P_{k-j}) + \sum_{j=1}^{N}(c_{2j/m}Q_{j+k} + c_{2j/m}Q_{j-k} - c_{2j/m}Q_{k-j})$$

$$\frac{\partial f_2^{10}}{\partial \dot{a}_{10}} = 3a_{20}b_{2k/m} + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}b_{2(i+k)} + b_{2i/m}b_{2(i-k)} + b_{2i/m}b_{2(k-i)}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}c_{2(i+k)} + c_{2i/m}c_{2(i-k)} - c_{2i/m}c_{2(k-i)})$$

$$\frac{\partial f_2^1}{\partial \dot{a}_{20}} = 3\frac{\partial f_2^2}{\partial \dot{a}_{10}},\ \frac{\partial f_2^2}{\partial \dot{a}_{20}} = 3\frac{\partial f_2^4}{\partial \dot{a}_{10}},\ \frac{\partial f_2^3}{\partial \dot{a}_{20}} = 2\frac{\partial f_2^5}{\partial \dot{a}_{10}},\ \frac{\partial f_2^4}{\partial \dot{a}_{20}} = 3\frac{\partial f_2^8}{\partial \dot{a}_{10}},\ \frac{\partial f_2^5}{\partial \dot{a}_{20}} = \frac{1}{2}\frac{\partial f_2^9}{\partial \dot{a}_{10}},\ \frac{\partial f_2^6}{\partial \dot{a}_{20}} = \frac{\partial f_2^{10}}{\partial \dot{a}_{10}},$$

$$\frac{\partial f_2^9}{\partial \dot{a}_{20}} = 2\dot{a}_{10}P_{k/m}$$

$$\frac{\partial f_3^2}{\partial \dot{a}_{10}} = \sum_{i=1}^{N}(B_{i/m}C_{i+k} + B_{i/m}C_{k-i} - B_{i/m}C_{i-k})$$

$$\frac{\partial f_3^4}{\partial \dot{a}_{10}} = \sum_{i=1}^{N}(B_{i/m}Q_{i+k} + B_{i/m}Q_{k-i} - B_{i/m}Q_{i-k}) + \sum_{i=1}^{N}(C_{i/m}P_{i-k} + C_{i/m}P_{k-i} - C_{i/m}P_{i+k})$$

$$\frac{\partial f_3^5}{\partial \dot{a}_{10}} = a_{20}C_{k/m} + \frac{1}{2}\sum_{i=1}^{N}(b_{2i/m}C_{i+k} + b_{2i/m}C_{k-i} - b_{2i/m}C_{i-k}) + \frac{1}{2}\sum_{i=1}^{N}(c_{2i/m}B_{k-i} + c_{2i/m}B_{i-k} - c_{2i/m}B_{i+k})$$

$$\frac{\partial f_3^8}{\partial \dot{a}_{10}} = 3\sum_{i=1}^{N}(P_{i/m}Q_{i+k} + P_{i/m}Q_{k-i} - P_{i/m}Q_{i-k})$$

$$\frac{\partial f_3^9}{\partial \dot{a}_{10}} = 2a_{20}Q_{k/m} + \sum_{i=1}^{N}(b_{2i/m}Q_{i+k} + b_{2i/m}Q_{k-i} - b_{2i/m}Q_{i-k}) + \sum_{i=1}^{N}(c_{2i/m}P_{k-i} + c_{2i/m}P_{i-k} - c_{2i/m}P_{i+k})$$

$$\frac{\partial f_3^{10}}{\partial \dot{a}_{10}} = 2a_{20}c_{2k/m} + \sum_{i=1}^{N}(b_{2i/m}c_{2(i+k)} + b_{2i/m}c_{2(k-i)} - b_{2i/m}c_{2(i-k)})$$

$$\frac{\partial f_3^1}{\partial \dot{a}_{20}} = 3\frac{\partial f_3^2}{\partial \dot{a}_{10}}, \frac{\partial f_3^2}{\partial \dot{a}_{20}} = \frac{\partial f_3^4}{\partial \dot{a}_{10}}, \frac{\partial f_3^3}{\partial \dot{a}_{20}} = 2\frac{\partial f_3^5}{\partial \dot{a}_{10}}, \frac{\partial f_3^4}{\partial \dot{a}_{20}} = \frac{1}{3}\frac{\partial f_3^8}{\partial \dot{a}_{10}}, \frac{\partial f_3^5}{\partial \dot{a}_{20}} = \frac{1}{2}\frac{\partial f_3^9}{\partial \dot{a}_{10}}, \frac{\partial f_3^6}{\partial \dot{a}_{20}} = \frac{\partial f_3^{10}}{\partial \dot{a}_{10}}$$

$$\frac{\partial P_n}{\partial \dot{b}_{1n}} = 1, \frac{\partial B_n}{\partial \dot{b}_{2n}} = 1, \frac{\partial Q_n}{\partial \dot{c}_{1n}} = 1, \frac{\partial C_n}{\partial \dot{c}_{2n}} = 1$$

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